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Infinite-Dimensional Optimization for Sustainable Urban Water Management: Application to the City of Sarh (Chad)

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Abstract

Reliable access to drinking water remains a major challenge for sub-Saharan African cities facing structural, economic, and climatic constraints. This article develops an infinite-dimensional optimization framework for urban water management over an infinite time horizon. The model is formulated within optimal control theory, with investment effort, demand, production, and service disruptions represented as continuous-time functions. A nonlinear production law captures diminishing returns, while outages and quality degradation enter directly into the utility function. The framework is applied to the city of Sarh (Chad) using calibrated scenario data consistent with local operating conditions. Numerical experiments are then conducted over a one-year horizon under a reproducible daily myopic policy, viewed as a finite-horizon approximation of the infinite-horizon benchmark. The results identify regimes of stock depletion, effort saturation, and cost sensitivity, and they show how the framework can support long-term planning in resource-constrained urban areas.

Keywords

Infinite-Dimensional Optimization, Urban Water Management, Optimal Control, Sarh, Numerical Simulation

1. Introduction

Water is a vital resource, indispensable for life, public health, agriculture, and economic stability. It plays a central role in the achievement of the United Nations Sustainable Development Goals (SDGs), particularly those relating to health (SDG 3), universal access to safe drinking water (SDG 6), food security (SDG 2), and poverty reduction (SDG 1). Sound water governance is therefore a global imperative [1] [2]. In developing countries, and particularly in Chad, access to safe drinking water remains a structural challenge. Secondary cities such as Sarh face rapid demographic growth, aging hydraulic infrastructure, high climatic variability, and chronic underfunding. Service interruptions and poor water quality contribute to distribution that is both unstable and inequitable [3] [4].

In light of these challenges, management approaches based solely on empirical practices or ad hoc interventions have shown their limitations. Sustainable planning requires a systemic perspective, supported by rigorous tools that make it possible to anticipate needs, model flows, and optimize resource allocation over the long term [5] [6]. This work contributes to this effort. Our objective is to propose a realistic mathematical model, based on infinite-dimensional optimization, to simulate and optimize water management in the urban context of Sarh. The framework captures not only the temporal dynamics of demand and production but also the cumulative effects of management decisions in a constrained and uncertain environment.

To this end, we develop a mathematical framework grounded in infinite-dimensional optimization. The model is structured as an infinite-horizon optimal control system. Unlike classical linear models, we introduce a nonlinear production function that reflects saturation effects in investment efforts, together with penalties associated with interruptions and degraded water quality.

In parallel, we rely on advanced analytical ideas to address the complexity of the nonlinear differential equations arising from the model. Recent developments in the Adomian decomposition method [7] and stochastic multi-season optimization applied to sesame cultivation [8] are particularly relevant to our approach.

The objective of this article is twofold:

- to develop a mathematical model of optimal water management in a resource-constrained urban context, explicitly incorporating realistic nonlinearities;
- to simulate, using illustrative data, the effects of management policies on user satisfaction and the costs incurred by the Chadian Water Company (STE).

In this context, it is essential to situate our contribution within the existing scientific literature. The next section reviews the state of the art on infinite-dimensional optimization methods and water management models, highlighting the gaps that our approach seeks to address.

2. State of the Art

The optimal management of natural resources over an infinite horizon in dynamic and uncertain settings has attracted growing attention in the scientific literature. Theoretical models based on infinite-dimensional optimization provide powerful mathematical tools for such problems, which require continuous-time decisions under dynamic systems with multiple constraints. This section reviews the mathematical foundations of the field together with current water-management applications and recent developments in the treatment of uncertainty and nonlinearity, before positioning our contribution.

2.1. Infinite-Dimensional Optimization: Theory and Methods

Infinite-dimensional optimization problems involve decision variables that are functions defined over continuous time intervals that may extend to infinity. These formulations are especially appropriate for systems described by differential equations when the objective is to maximize a time-dependent cost or utility functional.

The theoretical foundations rely mainly on the Pontryagin Maximum Principle and the Hamilton-Jacobi-Bellman (HJB) equation, which provide necessary and, in some cases, sufficient optimality conditions [9]. In infinite-dimensional settings, HJB theory has been developed for Banach and Hilbert spaces, providing a rigorous basis for continuous-time decision models under uncertainty [10] [11].

The performance of numerical methods in infinite dimensions can also be assessed through recent comparative studies of nonlinear optimization algorithms on benchmark functions, including Rosenbrock's function [12]. Such studies help adapt solvers more effectively to strongly constrained or unstable problems.

2.2. Models for Water Resource Management

The application of optimal control to water management has enabled researchers to go beyond empirical or purely deterministic models long used in water engineering. These approaches integrate renewal dynamics, variable demand, treatment costs, and infrastructural constraints. Classical, often linear, models such as those proposed by Loucks *et al.* [5] formed the basis of planning policies in many contexts. However, their ability to represent real systems in sufficient operational detail remains limited. More recent studies increasingly incorporate adaptive feedback, uncertainty, and operational constraints, thereby providing frameworks that are closer to operational decision-making.

2.3. Uncertainty, Interruptions, and Nonlinearities in Production

The complexity of water planning stems from uncertainty in climatic conditions, demand fluctuations, and infrastructure failures. Stochastic and risk-aware optimization frameworks have been developed to incorporate these sources of uncertainty explicitly [6] [13].

Many cities in the Global South experience service disruptions and deteriorating water quality, both of which reduce user satisfaction and operator performance. These aspects remain underrepresented in optimal-control models. The production function $r(u)=\frac{\beta u}{1+\delta u}$ captures diminishing returns from investment or management effort and is therefore more suitable for systems affected by saturation or marginal inefficiency.

Complementary references in optimal control emphasize viscosity-solution methods, semiconcavity, and practical control design frameworks [14]-[17]. On the water-management side, classic and modern studies have examined performance criteria, reservoir-operation reviews, simulation-optimization, metaheuristics, and predictive-control strategies for urban networks [18]-[25].

2.4. Positioning of Our Contribution

This work proposes both a conceptual and applied advance along four axes:

- The application of an infinite-dimensional optimization framework to a vulnerable urban water network, subject to interruptions and severe logistical constraints;
- The integration of a nonlinear production function, better suited to the reality of diminishing returns in disadvantaged urban areas;
- The explicit formalization of the impacts of service failures (frequency of outages and degraded quality) on the social utility function;
- The development of a robust simulator to evaluate alternative planning and decision-support scenarios in a context such as Sarh.

Within this framework, rigorous mathematical formalization is essential to describe the system's dynamic behavior, incorporate structural constraints, and explore optimal trajectories for effort allocation. The next section develops this framework around an infinite-dimensional optimization model.

3. Integrated Mathematical Framework and Application to Sarh

This section presents the infinite-horizon optimal-control benchmark together with the Sarh-specific parametrization used later in the numerical experiments. Merging the theoretical and applied ingredients avoids repeating the same mathematical structure twice and makes explicit which elements are generic and which are calibrated to the Sarh case.

3.1. Infinite-Horizon Benchmark Problem

We consider a dynamic system governed by the differential equation:

$$\dot{x}(t) = f\left(x(t), u(t), t\right), \quad x(0) = x_0 \geq 0, \quad t \geq 0,$$

where $x(t)$ represents the quantity of water available in the network at time t, and $u(t)$ denotes the investment or management effort.

In the context of water management, this optimal-control system incorporates climate forecasts, seasonal variations, and projected needs, thereby supporting infrastructure planning aimed at maximizing social benefit. It also provides a framework for cost control, quality monitoring, and long-term sustainability.

The objective is to maximize the functional:

$$J(u) = \int_0^\infty e^{-\rho t}\left[\alpha S\left(x(t), u(t), t\right) - c(t)u(t)\right]dt,$$

where $\rho > 0$ is a discount rate ensuring convergence of the integral [9], and $\alpha > 0$ is the social weight assigned to user satisfaction.

The control variable is subject to the admissibility constraint:

$$u \in \mathcal{U}_{ad},$$

where the admissible set is defined as

$$\mathcal{U}_{ad} := \left\{u \in L^2_{loc}(0,\infty) \mid 0 \leq u(t) \leq u_{max}\right\}.$$

Here L^2 is a Hilbert space equipped with the inner product

$$\langle u,v\rangle = \int_0^\infty u(t)v(t)\,\mathrm{d}t,$$

which guarantees weak compactness and projection properties [9] [11]. The state variable $x(t)$ is considered in the Banach space $C([0,\infty))$, suitable for handling discontinuities arising from service interruptions.

3.2. Pontryagin's Maximum Principle

The Hamiltonian is defined as:

$$\mathcal{H}(x,u,\lambda,t) = \mathrm{e}^{-\rho t}\left[\alpha S(x(t),u(t),t) - c(t)u(t)\right] + \lambda(t) f(x,u,t).$$

The associated adjoint system is:

$$\begin{cases} \dot{\lambda}(t) = -\dfrac{\partial \mathcal{H}}{\partial x}(x,u,\lambda,t) = -\alpha \mathrm{e}^{-\rho t}\dfrac{\partial S}{\partial x}(x,u,t) - \lambda(t)\dfrac{\partial f}{\partial x}(x,u,t), \\ u^*(t) = \arg\max\limits_{u\in[0,u_{\max}]} \mathcal{H}(x,u,\lambda,t). \end{cases}$$

This system provides the necessary conditions for any optimal pair (x^*,u^*) [9]. For the infinite-horizon benchmark, the usual transversality condition is imposed conceptually,

$$\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda(t)x(t) = 0,$$

which rules out trajectories that keep positive shadow-value-weighted stock at infinity.

3.3. Hamilton-Jacobi-Bellman Equation

An alternative characterization is obtained via the value function $V(x,t)$, defined as the maximal achievable utility starting from state x at time t. The HJB equation reads:

$$\rho V(x,t) = \max_{u\in[0,u_{\max}]}\left[\alpha S(x,u,t) - c(t)u + V'(x,t) f(x,u,t)\right],$$

where V is assumed differentiable and f sufficiently regular [10] [11].

Here, $V'(x,t)$ denotes the partial derivative of V with respect to x, and $f(x,u,t) = r(u) - d(t)$. This equation links the optimal value at each state to the trade-off between instantaneous utility and system dynamics.

3.4. Existence of an Optimal Solution

Under the following assumptions:

- $f(x,u,t)$ is Lipschitz continuous in x and continuous in u,
- $J(u)$ is upper semi-continuous,
- $\mathcal{U}_{\mathrm{ad}}$ is closed, bounded, and convex,
- $S(x,u,t)$ is continuous and bounded,

There exists an optimal solution $u^* \in \mathcal{U}_{\mathrm{ad}}$ maximizing $J(u)$. This result follows from classical theorems of weak compactness and weak convergence in Hilbert spaces [9].

3.5. Sarh-Specific Parametrization and Calibration

The town of Sarh is located in southern Chad on the banks of the Chari River. It is the country's fourth-largest city, after N'Djamena, Moundou, and Abéché. It covers an area of about 3000 hectares and comprises six districts. For the 2025 scenario used in this paper, we retain a population of 202,873 inhabitants.

The calibration combines measured or report-based inputs with scenario assumptions. Population and urban water-access context are taken from official country-data and WASH monitoring portals [3] [4]. Per-capita demand is fixed at 50 L/person/day, a planning value commonly used for basic urban-service scenarios in constrained settings; this yields the annual mean demand level of 10143.65 m^3/day. The seasonal amplitude 2028.73 and phase shift −3.37 in $d(t)$ are scenario parameters chosen to represent moderate intra-annual variability around that mean rather than a direct statistical fit. The initial stock $x_0 = 50000\ \mathrm{m}^3$ and the control bound $u_{\max} = 20000$ are engineering scenarios selected to test medium-scale storage and effort limits; they should therefore be read as assumed planning values, not as audited operational measurements.

Water management in Sarh is then formalized through the following stock dynamics:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = r(u(t)) - d(t),\ \ x(0) = x_0 \geq 0. \tag{1}$$

To capture realistically the diminishing returns associated with management effort in the urban context of Sarh, we specify water production through a nonlinear law.

Unlike the linear form $r(u) = \beta u$, we adopt a nonlinear formulation that reflects diminishing returns:

$$r(u) = \frac{\beta u}{1 + \delta u},\ \ \text{with } \beta > 0, \delta > 0. \tag{2}$$

This rational form captures the saturation effect: beyond a certain level of investment, marginal gains become negligible, which is consistent with the technical limits of the local hydraulic system.

3.6. Effects of Interruptions and Degraded Quality on Satisfaction

In the Sarh context, frequent interruptions ($\omega(t)$) and degraded water quality ($q(t)$) directly influence the perceived level of service. We introduce:

- $\omega(t) \in \{0,1\}$: effective service indicator (0 = interruption, 1 = service available);
- $q(t) \in [0,1]$: quality coefficient (1 = perfect quality, 0 = severely degraded quality).

The actual service thus becomes:

$$s(t) = S(x(t), u(t), t) \cdot \omega(t) \cdot q(t) \tag{3}$$

where

$$S(x(t), u(t), t) = \min\{x(t) + r(u(t)), d(t)\}$$

and the updated instantaneous utility is defined by

$$U(t) = e^{-\rho t}\left[\alpha \cdot s(t) - c(t) \cdot u(t)\right].\ \delta = 10^{-4} \tag{4}$$

where $c(t)$ is the unit cost per unit effort $u(t)$. This utility function is the core of the system's instantaneous performance, combining adjusted user satisfaction with the associated operational cost. We can now formulate the complete optimization problem rigorously while accounting for the system's dynamic, structural, and functional constraints.

3.7. Complete Formulation of the Problem with Constraints

We now have the elements needed to formulate the global problem of optimal water management in Sarh as a continuous-time, infinite-horizon optimal-control problem. The objective is to determine an admissible control policy $u(\cdot)$ that maximizes total discounted utility, while respecting the system's physical dynamics and operational constraints:

$$\max_{u(\cdot)\in\mathcal{U}_{ad}} J(u) = \int_0^\infty U(t)\,\mathrm{d}t, \tag{5}$$

under the following constraints:

$$\dot{x}(t) = \frac{\beta u(t)}{1+\delta u(t)} - d(t),\ \ x(0) = x_0 \ge 0, \tag{6}$$

$$s(t) = \gamma(t)\min\left\{x(t) + \frac{\beta u(t)}{1+\delta u(t)}, d(t)\right\}, \tag{7}$$

$$u(t) \in \mathcal{U}_{ad} \subset L^2_{\mathrm{loc}}(0,\infty), \tag{8}$$

where $\gamma(t) = \omega(t)\cdot q(t) \in [0,1]$ models quality of service through interruptions and degraded water quality. This formulation makes explicit:

- the nonlinear dynamics of production (diminishing returns),
- the direct effect of service failures on delivered service,
- the infinite-time integration specific to strategic planning.

The problem thus posed is well defined under reasonable assumptions on the functions $d(t)$, $c(t)$ and $\gamma(t)$ (continuity, boundedness, positivity), and admits an optimal solution according to the existence results presented above.

- Objective (5) maximizes discounted welfare by combining delivered service and operating cost.
- Constraint (6) describes the evolution of the water stock over time under demand pressure and diminishing returns from effort.
- Constraint (7) states that satisfaction cannot exceed either available supply or demand and is reduced by outages or poor-quality water through $\gamma(t)$.
- Constraint (8) restricts the control to realistic, measurable effort paths over long horizons.

4. Numerical Simulations

This section numerically illustrates the behavior of the Sarh water system under the model developed in Section 3. We use a synthetic yet representative one-year scenario to study how management policies shape the water stock, effective service, and social utility.

4.1. Numerical Policy Class and Finite-Horizon Approximation

The infinite-horizon formulation above is the theoretical benchmark of the paper. The numerical experiments below do *not* solve the full Pontryagin boundary-value problem or the HJB equation on an infinite domain. Instead, they implement a finite-horizon daily myopic policy over $T = 365$ days. At each day t, the control is selected on a discrete grid $\{0, 100, \cdots, u_{\max}\}$ by maximizing the one-step payoff computed with the expected service-quality factor:

$$u_t \in \arg\max_{u \in [0, u_{\max}]} \left\{ \alpha \mathbb{E}[\gamma_t] \min\left(d_t, x_t + r(u)\right) - c_t u \right\}.$$

The stock is then updated one day forward. The one-year experiment is therefore a truncated approximation of the infinite-horizon model with zero terminal continuation value, that is, $V_T(x) = 0$ at the end of the simulated year. Conceptually, this finite-horizon truncation is consistent with the infinite-horizon transversality condition stated above, but it should be interpreted as an operational policy approximation rather than as the exact infinite-horizon optimum.

4.2. Data, Stochastic Assumptions, and Reproducibility

We consider one year ($T = 365$) with daily time step ($\Delta t = 1$ day). Parameters:

- Population: 202,873 inhabitants;
- Per-capita demand: 0.05 m^3/person/day (50 L/day);
- Seasonal baseline demand:

$$d(t) = 10143.65 + 2028.73 \sin(0.172t - 3.37) \left[\mathrm{m}^3/\mathrm{day} \right];$$

- Discount rate: $\rho = 0.0005$ per day (≈0.1825 annually);
- Satisfaction weight: $\alpha = 1$ unless stated otherwise;
- Production with saturation: $r(u) = \dfrac{\beta u}{1 + \delta u}$ with $\beta = 1$, ;
- Unit cost (baseline): $c = 0.0008$ per unit of u per day;
- Control bounds: $u(t) \in [0, u_{\max}]$, $u_{\max} = 20000$;
- Initial stock: $x(0) = 50000\ \mathrm{m}^3$.

In the *baseline*, service quality is exogenous: $\gamma(t)$ is drawn independently each day from a three-point distribution,

$$\gamma(t) = \begin{cases} 0 & \text{with probability } 0.25, \\ 0.7 & \text{with probability } 0.30, \\ 1 & \text{with probability } 0.45. \end{cases}$$

Thus, outages and quality degradation are not directly controlled by contemporaneous investment $u(t)$ in the simulations; they enter as external disturbances. In the final experiment below, we instead use $\gamma(t) = 0.8 + 0.2 Z_t$ with $Z_t \sim \mathrm{Beta}(5, 2)$, again independently across days. All stochastic trajectories reported in this paper are generated with a fixed random seed equal to 2025, and the tables and figures correspond to that reproducible seeded run rather than to Monte Carlo averages. Delivered service is

$$s(t) = \gamma(t)\min\{d(t), x(t) + r(u(t))\},$$

and the stock updates accordingly. The production capacity at the bound is $r(u_{\max}) \approx 6667\ \mathrm{m}^3/\mathrm{day}$.

4.3. Baseline Dynamics

With $\alpha = 1$ and baseline costs, the objective reduces to quality-adjusted volume minus daily production costs. The total discounted utility is

$$J_{\text{baseline}} = 1457433.25.$$

Figure 1 shows that the stock is rapidly depleted in the first days due to sustained demand; thereafter the system operates in a production-limited regime. Interruptions/degraded quality appear as troughs in $s(t)$. Because unit cost is

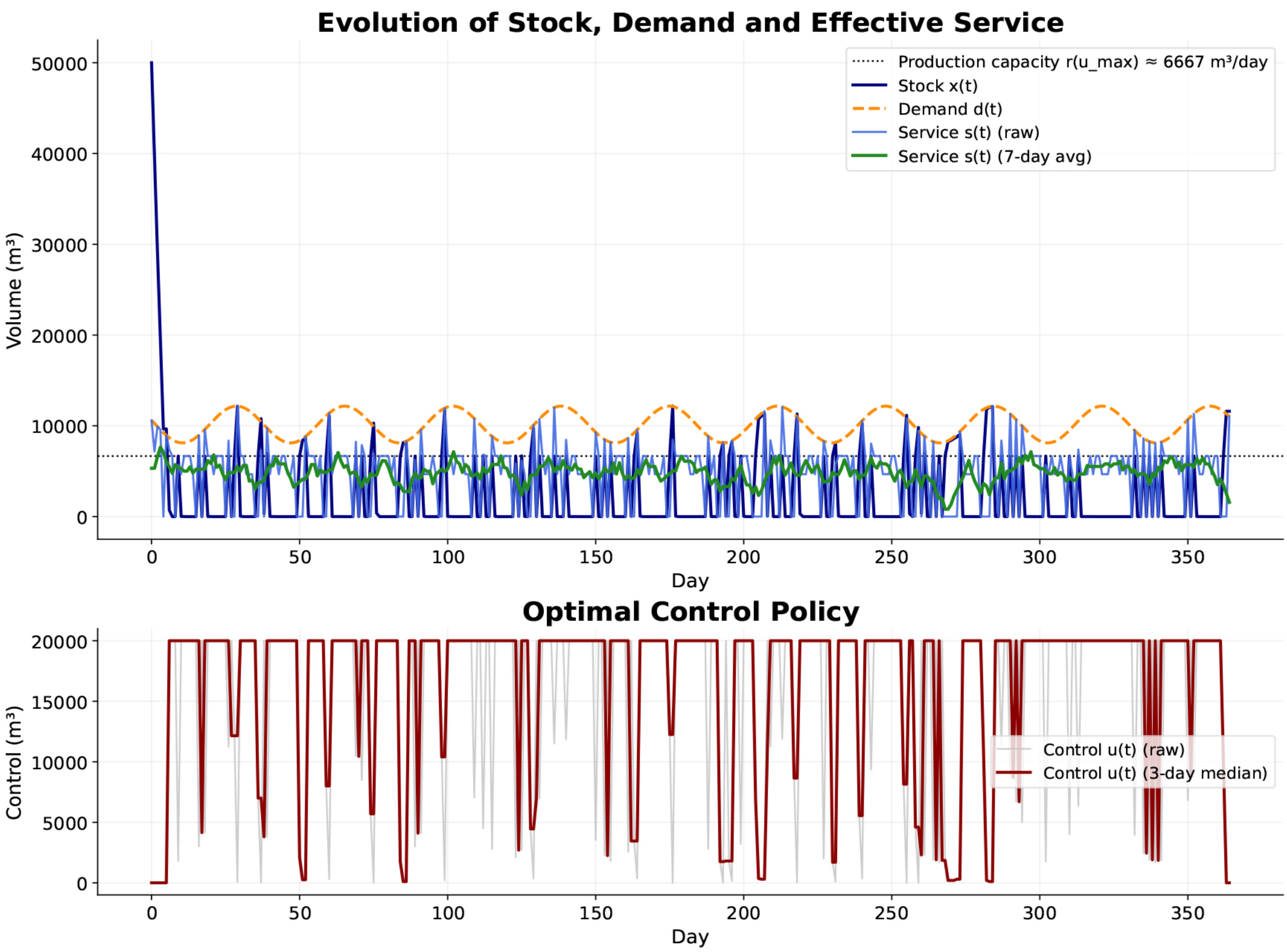

Figure 1. Baseline dynamics. *Top*: stock $x(t)$ (navy), demand $d(t)$ (orange, dashed), effective service $s(t)$ (light blue, raw) and 7-day average (dark green); dotted line: $r(u_{\max}) \approx 6667\ \mathrm{m}^3/\mathrm{day}$. *Bottom*: optimal control $u(t)$ (3-day median). The stock falls quickly and the system becomes production-limited; service dips reflect interruptions/quality degradation.

small, the control frequently operates near $u_{\max}$. This recurrent saturation is mechanically explained by the one-step objective: with the present calibration, the marginal benefit $\alpha\mathbb{E}[\gamma_t]r'(u)$ exceeds the marginal cost c over most of the admissible range, so the policy continues to increase effort until it reaches the bound. The saturation in $r(u)$ softens the gains from very large effort but does not offset the low baseline unit cost.

4.4. Effect of a 10% Cost Increase

We increase the unit cost by 10% ($c \mapsto 1.1c$). The discounted utility becomes

$$J_{+10\%} = 1457050.33,\ \ \Delta J = -382.92\ \ (-0.0263\%).$$

Trajectories of stock and service remain close to baseline (**Figure 2**); the small loss in J is consistent with a high weight on service when $\alpha = 1$.

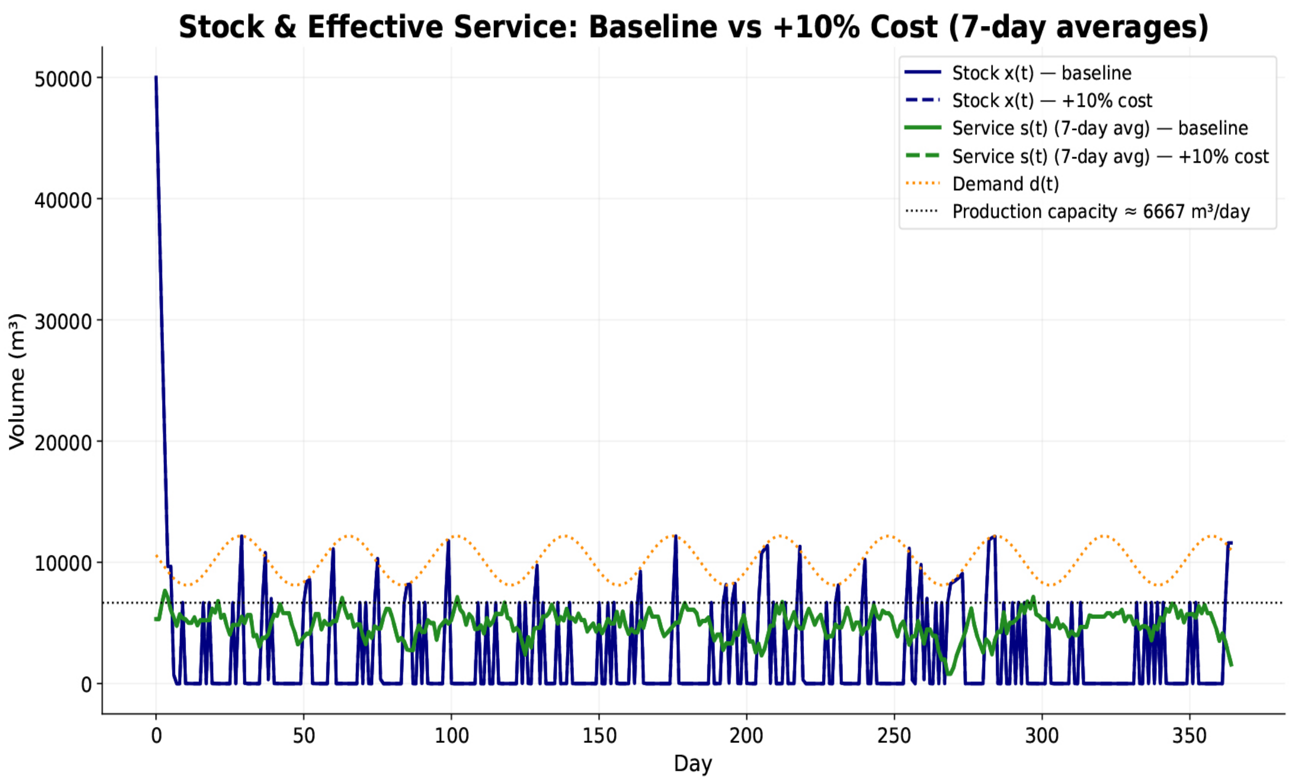

Figure 2. Baseline vs. +10% cost. Seven-day averages of $x(t)$ and $s(t)$ under baseline and +10% cost. Differences are modest because, with $\alpha = 1$, service value dominates cost.

4.5. Cost Sensitivity

We vary the unit cost by factors $\{1,2,5,10,50,100\}$ of the baseline $c = 0.0008$, keeping the same daily myopic policy with expected γ. **Table 1** and **Figure 3** summarize: (a) total discounted utility J; (b) average daily control $\bar{u}$; (c) representative 3-day-median control paths for factors 1, 10, 100. Utility is relatively flat up to 10× and then erodes, with a sharp drop at 100×; average effort remains near the maximum until costs become extreme.

Table 1. Cost sensitivity under daily myopic control: total discounted utility and control statistics.

Factor	Unit cost	Total discounted utility J	$\bar{u}$ (m^3/day)	Median u
1	0.0008	1,457,433.25	14,356.16	20,000
2	0.0016	1,453,604.02	14,356.16	20,000
5	0.0040	1,442,116.32	14,356.16	20,000
10	0.0080	1,422,970.16	14,356.16	20,000
50	0.0400	1,269,808.87	14,356.16	20,000
100	0.0800	1,078,381.69	14,247.67	19,600

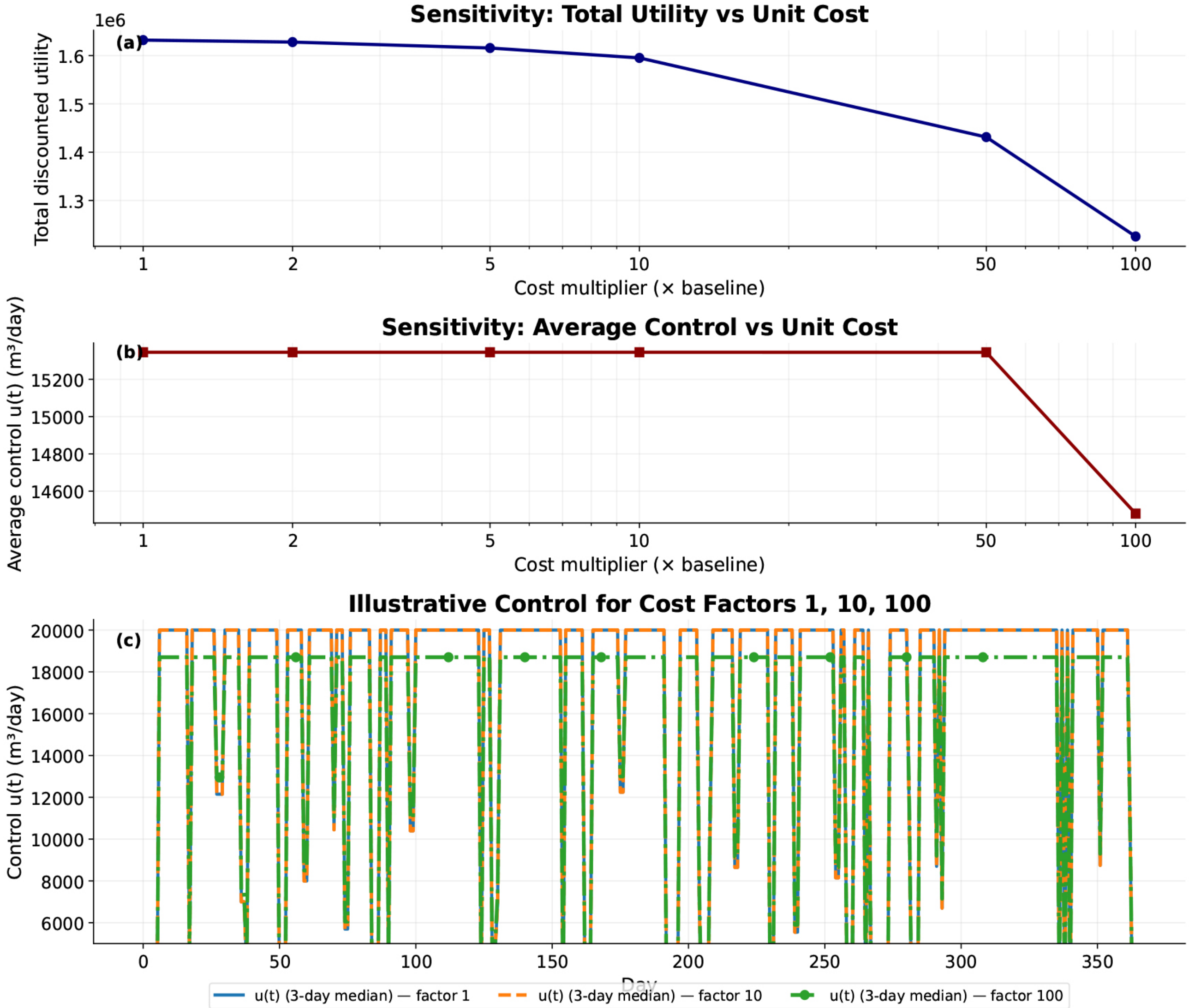

Figure 3. Cost sensitivity. (a) Total discounted utility vs. cost multiplier (log scale). (b) Average control $\bar{u}$. (c) Illustrative control paths (3-day medians) for factors 1, 10, 100. Performance is robust to moderate cost increases; utility and effort deteriorate only at very high costs.

4.6. High Cost with Imperfect Satisfaction and Realistic Quality

We finally consider a more realistic user response and quality profile: set $\alpha = 0.6$

and draw $\gamma(t) \in [0.8, 1.0]$ from a Beta-based distribution skewed toward high quality (no full outages). With a high unit cost (100× baseline), effort collapses and the system quickly becomes production-limited; despite acceptable quality, the lower satisfaction weight and high cost depress delivered service (Figure 4).

Figure 4. High cost, imperfect satisfaction, realistic quality. *Top*: $x(t)$, $d(t)$, and $s(t)$ (raw and 7-day average) with capacity line $r(u_{\max})$. The stock depletes early and the system becomes production-limited. *Bottom*: 3-day median control showing a sustained collapse in effort relative to baseline.

5. Discussion and Analysis

Taken together, the numerical experiments in Section 4 (Figures 1-4) reveal a consistent pattern: after early stock depletion, the system becomes production-limited; small cost perturbations leave outcomes largely unchanged, whereas extreme costs and imperfect satisfaction markedly depress service and effort. In what follows, we interpret these mechanisms, contrast the nonlinear formulation with linear alternatives, and derive implications for planning under uncertainty and operational constraints.

Linear vs. nonlinear production.

Using $r(u) = \beta u / (1 + \delta u)$ captures diminishing returns that are typical in constrained hydraulic systems. Unlike linear models ($r(u) = \kappa u$), the nonlinear form discourages unrealistically large efforts and produces more economically plausible solutions.

Impact of unit cost.

A dual regime emerges: (i) for low-moderate costs, optimal effort remains near $u_{\max}$ and J decreases only slightly; (ii) at very high costs (≥50×), J drops sharply and the average effort declines to contain expenditures.

Why the effort saturates.

Under the myopic policy, the control compares the marginal value of additional delivered water with the marginal operating cost. In stock-constrained periods, the relevant signal is approximately $\alpha \mathbb{E}[\gamma_t] r'(u) = \alpha \mathbb{E}[\gamma_t] \beta / (1 + \delta u)^2$. With the baseline calibration, this marginal value stays above c for most feasible u , so the computed policy keeps selecting values close to $u_{\max}$. A practical implication is that pure infrastructure effort dominates when the ratio c/α is low and continuity losses are moderate, whereas improving reliability or water quality becomes more attractive once outages are frequent enough to reduce the effective value of an extra unit of production.

Role of service quality.

When $\gamma(t) = 1$, service depends almost exclusively on production and stock. With fluctuating $\gamma(t)$, even maximal effort cannot guarantee high service: satisfaction drops and overproduction can be wasteful—motivating stochastic planning that internalizes outage probabilities.

Stock and dynamics.

A large initial stock (50,000 m^3) cushions seasonal oscillations when costs are low. Under high costs and lower α , the stock becomes critically low early, and the system remains production-limited thereafter.

Limitations and extensions.

1) Simulated data: integrating STE operational records and sensor series would strengthen external validity.

2) No spatial detail: Sarh is treated as a single node; a networked, multi-district model would capture heterogeneity.

3) Utility specification: a concave/saturating satisfaction function could be explored.

Future work includes multi-agent formulations (users-STE-regulator), stochastic demand and quality processes, and adaptive or MPC-style control with learning.

6. Conclusion and Recommendations

This paper developed a rigorous infinite-dimensional optimization framework for sustainable urban water management and applied it to the city of Sarh (Chad). By combining optimal-control theory with a nonlinear production technology and explicit penalties for outages and degraded quality, we demonstrated how management policies shape stocks, service, and welfare over long horizons. The theoretical model is paired with a reproducible finite-horizon daily myopic policy for simulation. Numerical experiments (**Figures 1-4**) revealed a consistent pattern: rapid early stock depletion leads to a production-limited regime; modest cost shocks have limited effects, whereas extreme costs and imperfect satisfaction substantially depress service and effort.

Summary of Theoretical and Practical Contributions

Theoretical.

- We formulated an infinite-horizon optimal control problem in a suitable functional setting (L^2 for controls, $C([0,\infty))$ for states) with realistic operational constraints.
- We employed a nonlinear production function $r(u)=\frac{\beta u}{1+\delta u}$ to capture diminishing returns to effort and prevent implausible bang-bang solutions.
- We integrated interruptions and quality degradation directly into the utility, linking reliability and perceived service to welfare.

Practical.

- We quantified how outages/quality shocks propagate to stocks and effective service.
- We characterized cost sensitivity of total discounted utility and effort, identifying robust and fragile operating zones.
- We highlighted the role of initial storage and quality in sustaining service under budget constraints.

Recommendations for STE and the Municipality of Sarh

- **Targeted effort thresholds.** Calibrate production around data-driven thresholds (where marginal utility equals marginal cost) to avoid unnecessary expenditure while protecting service.
- **Minimum stock buffer.** Adopt a simple rule (e.g., a seasonal buffer sized to demand amplitude) to mitigate early depletion and reduce reliance on costly peak effort.
- **Reduce outage exposure.** Coordinate with the power utility to prioritize pumping windows and backup supply at critical nodes; track a continuity KPI (e.g., % days without interruption).
- **Quality-aware operations.** Integrate turbidity/residual-chlorine triggers into dispatch so that effort is not wasted when $\gamma(t)$ is low; monitor an energy-per-volume KPI (kWh/m^3).
- **Planning toolchain.** Institutionalize the simulator as a decision-support tool for multi-year scenarios (cost shocks, demand growth, drought years), with reproducible inputs and versioned policies.

Research Perspectives

- **Empirical calibration.** Incorporate STE operational records, sensor streams, and surveys; estimate parameters (e.g., δ, cost) via Bayesian or likelihood-based calibration.
- **Spatial extension.** Move from a single-node model to a networked, district-level formulation with hydraulic/pressure constraints and equity targets.

- **Adaptive control.** Compare robust/MPC policies with ramping constraints and quality-aware penalties; learn disturbance models for demand and $\gamma(t)$ online.
- **Policy design.** Study tariff instruments and targeted subsidies under affordability constraints, and assess welfare-revenue trade-offs.
- **Interdisciplinary implementation.** Develop the model jointly with civil engineers, hydrologists, and public-economics specialists so that operational constraints and investment rules remain realistic.

Overall, the framework provides a transparent and extensible basis for long-term planning in resource-constrained cities. Its emphasis on reliability and quality, diminishing-returns technology, and explicit cost-benefit trade-offs makes it well suited for guiding investment and operational strategies toward the SDG 6 objectives.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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