Applied Mathematics, 2026, 17(1), 1-15
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Braille Code, Symmetry, Graph, Group, and Equivalence Relation

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How to cite this paper: Mareco, H.R.O. (2026) Braille Code, Symmetry, Graph, Group, and Equivalence Relation. *Applied Mathematics*, **17**, 1-15.
https://doi.org/10.4236/am.2026.171001

Received: October 26, 2025
Accepted: January 17, 2026
Published: January 20, 2026

Abstract

This work synthesises Braille code via elementary concepts of naive set theory, symmetry, graph, group, and equivalence relation. Thus, the Braille code can also be understood as an even more mathematically valuable teaching-learning resource.

Keywords

Braille Code, Set Theory, Symmetry, Teaching-Learning Resource

1. Introduction

This article will link very important things in themselves: the Braille code, naive set theory, symmetry, graph, group, and equivalence relation. In 1824, Louis Braille invented a set of symbols that could serve as mediators in the communication of information. This code was a solution to the total blindness of the Braille self, which began when he was a child due to an accident in one eye and was aggravated by infection in both eyes. Until today, his code is still used worldwide. This set is now known as the Braille Code. Initially, this system was designed for tactile reading, suitable for people who are blind or have impaired eyes. Symmetry, graph, group, and equivalence relation are basic mathematical concepts and remain relevant in its applications in all branches of Sciences. The primary goal of this paper is to transform Braille into a teaching-learning resource for the aforementioned mathematical concepts. Conversely, the paper suggests that the Braille code can be used as an example of the application of these mathematical concepts. How to do it? Let's consider the next statements.

"***Mathematics*** *is the art of giving the same name to different things.*"

"*A thing is* ***symmetrical*** *if one can subject it to a certain operation and it appears exactly the same after the operation.*"

Attributed, respectively, to Henri Poincaré and Hermann Weyl.

2. Braille Set

Here we apply a procedure that allows us to interpret the Braille code as a set. It is well known that the Braille code is a raised dot system that can be read by the fingers. Any Braille code's symbol is of the form

$$\begin{matrix} b_1 & b_4 \\ b_2 & b_5 \\ b_3 & b_6 \end{matrix}. \tag{2.1}$$

Each place of the Braille symbol may be empty or busy. Here, we associate to the Braille code the set

$$\mathcal{B}. \tag{2.2}$$

Any Braille symbol may be represented by the Braille cell defined in the form

$$x = \boxed{\begin{matrix} x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_6 \end{matrix}} = \overbrace{\boxed{\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}}}^{\alpha}\overbrace{\boxed{\begin{matrix} x_4 \\ x_5 \\ x_6 \end{matrix}}}^{\mu} = \alpha\mu. \tag{2.3}$$

Every symbol $x_{i=1,2,\cdots,6}$ is a space unit. The possible states of any space unit x_i are the following

$$x_i = \begin{cases} + & \text{raised dot} \\ 0 & \text{flat dot} \\ - & \text{hollow dot} \end{cases}. \tag{2.4}$$

If we like to read a Braille symbol $\begin{matrix} b_1 & b_4 \\ b_2 & b_5 \\ b_3 & b_6 \end{matrix}$ we will move our finger from the left to the right direction writing $\boxed{\begin{matrix} x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_6 \end{matrix}}$ such that $x_i = +$ if b_i is busy or that $x_i = 0$ if b_i is empty that means

$$\underbrace{\begin{matrix} b_1 & b_4 \\ b_2 & b_5 \\ b_3 & b_6 \end{matrix}}_{\substack{\rightarrow \\ \text{finger reading sense}}} \mapsto \underbrace{\boxed{\begin{matrix} x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_6 \end{matrix}}}_{\substack{\rightarrow \\ \text{reading sense}}} \quad x_i = \begin{cases} + & \text{if } b_i \text{ is busy} \\ 0 & \text{if } b_i \text{ is empty} \end{cases} \tag{2.5}$$

Writing in braille (using a stylus to push dots into paper, for example) requires a bit of efforts,

$$\underbrace{\begin{matrix} b_1 & b_4 \\ b_2 & b_5 \\ b_3 & b_6 \end{matrix}}_{\substack{\rightarrow \\ \text{finger reading sense}}} \mapsto \underbrace{\boxed{\begin{matrix} y_1 & y_4 \\ y_2 & y_5 \\ y_3 & y_6 \end{matrix}}}_{\substack{\leftarrow \\ \text{writing sense}}} \quad \begin{matrix} y_{i=4,5,6} = \begin{cases} - & \text{if } b_{i-3} \text{ is busy} \\ 0 & \text{if } b_{i-3} \text{ is empty} \end{cases} \\ y_{i=1,2,3} = \begin{cases} - & \text{if } b_{i+3} \text{ is busy} \\ 0 & \text{if } b_{i+3} \text{ is empty} \end{cases} \end{matrix} \tag{2.6}$$

Therefore there is the relation between reading and writing is given by

$$\underbrace{\begin{array}{|cc|}\hline y_1 & y_4\\ y_2 & y_5\\ y_3 & y_6\\ \hline\end{array}}_{\substack{\leftarrow\\ \text{writing sense}}} \mapsto \underbrace{\begin{array}{|cc|}\hline x_1 & x_4\\ x_2 & x_5\\ x_3 & x_6\\ \hline\end{array}}_{\substack{\rightarrow\\ \text{reading sense}}} \quad \begin{array}{l} x_{i=1,2,3}=\begin{cases}+ & \text{if } y_{i+3}=-\\ 0 & \text{if } y_{i+3}=0\end{cases}\\ x_{i=4,5,6}=\begin{cases}+ & \text{if } y_{i-3}=-\\ 0 & \text{if } y_{i-3}=0\end{cases}\end{array} \tag{2.7}$$

$$\underbrace{\begin{array}{|cc|}\hline x_1 & x_4\\ x_2 & x_5\\ x_3 & x_6\\ \hline\end{array}}_{\substack{\rightarrow\\ \text{reading sense}}} \mapsto \underbrace{\begin{array}{|cc|}\hline y_1 & y_4\\ y_2 & y_5\\ y_3 & y_6\\ \hline\end{array}}_{\substack{\leftarrow\\ \text{writing sense}}} \quad \begin{array}{l} y_{i=4,5,6}=\begin{cases}- & \text{if } x_{i-3}=+\\ 0 & \text{if } x_{i-3}=0\end{cases}\\ y_{i=1,2,3}=\begin{cases}- & \text{if } x_{i+3}=+\\ 0 & \text{if } x_{i+3}=0\end{cases}\end{array} \tag{2.8}$$

How many elements may have the Braille code $\mathcal{B}$? The unit of space x_1 may have two states. For the remains unit of space the number of possible states are the same, 2. Two cells $x, y \in \mathcal{B}$ are different if they have two space units x_i, y_i that are different,

$$\begin{array}{|cc|}\hline x_1 & x_4\\ x_2 & x_5\\ x_3 & x_6\\ \hline\end{array} \neq \begin{array}{|cc|}\hline y_1 & y_4\\ y_2 & y_5\\ y_3 & y_6\\ \hline\end{array} \Leftrightarrow \exists i; x_i \neq y_i\,. \tag{2.9}$$

Hence the total number of Braille cells (see Slate 1) is

$$(2\times 2\times 2)\times(2\times 2\times 2)=2^6=64\,. \tag{2.10}$$

$$\begin{array}{cccccccc}
\boxed{\begin{matrix}0&+\\0&+\\0&+\end{matrix}} & \boxed{\begin{matrix}+&+\\0&+\\0&+\end{matrix}} & \boxed{\begin{matrix}0&+\\+&+\\0&+\end{matrix}} & \boxed{\begin{matrix}0&+\\0&+\\+&+\end{matrix}} & \boxed{\begin{matrix}+&+\\+&+\\0&+\end{matrix}} & \boxed{\begin{matrix}+&+\\0&+\\+&+\end{matrix}} & \boxed{\begin{matrix}0&+\\+&+\\+&+\end{matrix}} & \boxed{\begin{matrix}+&+\\+&+\\+&+\end{matrix}}\\
\boxed{\begin{matrix}0&0\\0&+\\0&+\end{matrix}} & \boxed{\begin{matrix}+&0\\0&+\\0&+\end{matrix}} & \boxed{\begin{matrix}0&0\\+&+\\0&+\end{matrix}} & \boxed{\begin{matrix}0&0\\0&+\\+&+\end{matrix}} & \boxed{\begin{matrix}+&0\\+&+\\0&+\end{matrix}} & \boxed{\begin{matrix}+&0\\0&+\\+&+\end{matrix}} & \boxed{\begin{matrix}0&0\\+&+\\+&+\end{matrix}} & \boxed{\begin{matrix}+&0\\+&+\\+&+\end{matrix}}\\
\boxed{\begin{matrix}0&+\\0&0\\0&+\end{matrix}} & \boxed{\begin{matrix}+&+\\0&0\\0&+\end{matrix}} & \boxed{\begin{matrix}0&+\\+&0\\0&+\end{matrix}} & \boxed{\begin{matrix}0&+\\0&0\\+&+\end{matrix}} & \boxed{\begin{matrix}+&+\\+&0\\0&+\end{matrix}} & \boxed{\begin{matrix}+&+\\0&0\\+&+\end{matrix}} & \boxed{\begin{matrix}0&+\\+&0\\+&+\end{matrix}} & \boxed{\begin{matrix}+&+\\+&0\\+&+\end{matrix}}\\
\boxed{\begin{matrix}0&+\\0&+\\0&0\end{matrix}} & \boxed{\begin{matrix}+&+\\0&+\\0&0\end{matrix}} & \boxed{\begin{matrix}0&+\\+&+\\0&0\end{matrix}} & \boxed{\begin{matrix}0&+\\0&+\\+&0\end{matrix}} & \boxed{\begin{matrix}+&+\\+&+\\0&0\end{matrix}} & \boxed{\begin{matrix}+&+\\0&+\\+&0\end{matrix}} & \boxed{\begin{matrix}0&+\\+&+\\+&0\end{matrix}} & \boxed{\begin{matrix}+&+\\+&+\\+&0\end{matrix}}\\
\boxed{\begin{matrix}0&0\\0&0\\0&+\end{matrix}} & \boxed{\begin{matrix}+&0\\0&0\\0&+\end{matrix}} & \boxed{\begin{matrix}0&0\\+&0\\0&+\end{matrix}} & \boxed{\begin{matrix}0&0\\0&0\\+&+\end{matrix}} & \boxed{\begin{matrix}+&0\\+&0\\0&+\end{matrix}} & \boxed{\begin{matrix}+&0\\0&0\\+&+\end{matrix}} & \boxed{\begin{matrix}0&0\\+&0\\+&+\end{matrix}} & \boxed{\begin{matrix}+&0\\+&0\\+&+\end{matrix}}\\
\boxed{\begin{matrix}0&0\\0&+\\0&0\end{matrix}} & \boxed{\begin{matrix}+&0\\0&+\\0&0\end{matrix}} & \boxed{\begin{matrix}0&0\\+&+\\0&0\end{matrix}} & \boxed{\begin{matrix}0&0\\0&+\\+&0\end{matrix}} & \boxed{\begin{matrix}+&0\\+&+\\0&0\end{matrix}} & \boxed{\begin{matrix}+&0\\0&+\\+&0\end{matrix}} & \boxed{\begin{matrix}0&0\\+&+\\+&0\end{matrix}} & \boxed{\begin{matrix}+&0\\+&+\\+&0\end{matrix}}\\
\boxed{\begin{matrix}0&+\\0&0\\0&0\end{matrix}} & \boxed{\begin{matrix}+&+\\0&0\\0&0\end{matrix}} & \boxed{\begin{matrix}0&+\\+&0\\0&0\end{matrix}} & \boxed{\begin{matrix}0&+\\0&0\\+&0\end{matrix}} & \boxed{\begin{matrix}+&+\\+&0\\0&0\end{matrix}} & \boxed{\begin{matrix}+&+\\0&0\\+&0\end{matrix}} & \boxed{\begin{matrix}0&+\\+&0\\+&0\end{matrix}} & \boxed{\begin{matrix}+&+\\+&0\\+&0\end{matrix}}\\
\boxed{\begin{matrix}0&0\\0&0\\0&0\end{matrix}} & \boxed{\begin{matrix}+&0\\0&0\\0&0\end{matrix}} & \boxed{\begin{matrix}0&0\\+&0\\0&0\end{matrix}} & \boxed{\begin{matrix}0&0\\0&0\\+&0\end{matrix}} & \boxed{\begin{matrix}+&0\\+&0\\0&0\end{matrix}} & \boxed{\begin{matrix}+&0\\0&0\\+&0\end{matrix}} & \boxed{\begin{matrix}0&0\\+&0\\+&0\end{matrix}} & \boxed{\begin{matrix}+&0\\+&0\\+&0\end{matrix}}
\end{array}$$

Slate 1. All 64 possible Braille cells of the Braille set $\mathcal{B}$.

Braille set $\mathcal{B}$ may be viewed like a Cartesian product of the set

$$\mathcal{R}=\left\{\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}+\\0\\0\end{bmatrix},\begin{bmatrix}0\\+\\0\end{bmatrix},\begin{bmatrix}0\\0\\+\end{bmatrix},\begin{bmatrix}+\\+\\0\end{bmatrix},\begin{bmatrix}+\\0\\+\end{bmatrix},\begin{bmatrix}0\\+\\+\end{bmatrix},\begin{bmatrix}+\\+\\+\end{bmatrix}\right\}. \tag{2.11}$$

According to this set we obtain

$$\mathcal{B}=\{\alpha\mu;\alpha,\mu\in\mathcal{R}\}. \tag{2.12}$$

Let's introduce a function - called swap "ˆ" from $\{+,0\}$ into $\{+,0\}$ in the following way:

$$\begin{matrix}\hat{+} & = & 0\\ \hat{0} & = & +\end{matrix}. \tag{2.13}$$

The swap can use to give us the next function from $\mathcal{R}$ into itself

$$\begin{bmatrix}u\\v\\w\end{bmatrix}\mapsto\widetilde{\begin{bmatrix}u\\v\\w\end{bmatrix}}=\begin{bmatrix}\hat{u}\\\hat{v}\\\hat{w}\end{bmatrix}. \tag{2.14}$$

If we define

$$\mathcal{A}=\left\{\mathbf{1}=\begin{bmatrix}+\\+\\0\end{bmatrix},\mathbf{2}=\begin{bmatrix}+\\0\\+\end{bmatrix},\mathbf{3}=\begin{bmatrix}0\\+\\+\end{bmatrix},\mathbf{4}=\begin{bmatrix}+\\+\\+\end{bmatrix}\right\} \tag{2.15}$$

Braille cells

$$\mathcal{B}=\{\alpha\mu;\alpha\text{ or }\tilde{\alpha},\mu\text{ or }\tilde{\mu}\in\mathcal{A}\} \tag{2.16}$$

can be represented as shown in the Slate 2.

$\tilde{\mathbf{4}}\mathbf{4}$	$\tilde{\mathbf{3}}\mathbf{4}$	$\tilde{\mathbf{2}}\mathbf{4}$	$\tilde{\mathbf{1}}\mathbf{4}$	$\mathbf{1}\mathbf{4}$	$\mathbf{2}\mathbf{4}$	$\mathbf{3}\mathbf{4}$	$\mathbf{4}\mathbf{4}$
$\tilde{\mathbf{4}}\mathbf{3}$	$\tilde{\mathbf{3}}\mathbf{3}$	$\tilde{\mathbf{2}}\mathbf{3}$	$\tilde{\mathbf{1}}\mathbf{3}$	$\mathbf{1}\mathbf{3}$	$\mathbf{2}\mathbf{3}$	$\mathbf{3}\mathbf{3}$	$\mathbf{4}\mathbf{3}$
$\tilde{\mathbf{4}}\mathbf{2}$	$\tilde{\mathbf{3}}\mathbf{2}$	$\tilde{\mathbf{2}}\mathbf{2}$	$\tilde{\mathbf{1}}\mathbf{2}$	$\mathbf{1}\mathbf{2}$	$\mathbf{2}\mathbf{2}$	$\mathbf{3}\mathbf{2}$	$\mathbf{4}\mathbf{2}$
$\tilde{\mathbf{4}}\mathbf{1}$	$\tilde{\mathbf{3}}\mathbf{1}$	$\tilde{\mathbf{2}}\mathbf{1}$	$\tilde{\mathbf{1}}\mathbf{1}$	$\mathbf{1}\mathbf{1}$	$\mathbf{2}\mathbf{1}$	$\mathbf{3}\mathbf{1}$	$\mathbf{4}\mathbf{1}$
$\tilde{\mathbf{4}}\tilde{\mathbf{1}}$	$\tilde{\mathbf{3}}\tilde{\mathbf{1}}$	$\tilde{\mathbf{2}}\tilde{\mathbf{1}}$	$\tilde{\mathbf{1}}\tilde{\mathbf{1}}$	$\mathbf{1}\tilde{\mathbf{1}}$	$\mathbf{2}\tilde{\mathbf{1}}$	$\mathbf{3}\tilde{\mathbf{1}}$	$\mathbf{4}\tilde{\mathbf{1}}$
$\tilde{\mathbf{4}}\tilde{\mathbf{2}}$	$\tilde{\mathbf{3}}\tilde{\mathbf{2}}$	$\tilde{\mathbf{2}}\tilde{\mathbf{2}}$	$\tilde{\mathbf{1}}\tilde{\mathbf{2}}$	$\mathbf{1}\tilde{\mathbf{2}}$	$\mathbf{2}\tilde{\mathbf{2}}$	$\mathbf{3}\tilde{\mathbf{2}}$	$\mathbf{4}\tilde{\mathbf{2}}$
$\tilde{\mathbf{4}}\tilde{\mathbf{3}}$	$\tilde{\mathbf{3}}\tilde{\mathbf{3}}$	$\tilde{\mathbf{2}}\tilde{\mathbf{3}}$	$\tilde{\mathbf{1}}\tilde{\mathbf{3}}$	$\mathbf{1}\tilde{\mathbf{3}}$	$\mathbf{2}\tilde{\mathbf{3}}$	$\mathbf{3}\tilde{\mathbf{3}}$	$\mathbf{4}\tilde{\mathbf{3}}$
$\tilde{\mathbf{4}}\tilde{\mathbf{4}}$	$\tilde{\mathbf{3}}\tilde{\mathbf{4}}$	$\tilde{\mathbf{2}}\tilde{\mathbf{4}}$	$\tilde{\mathbf{1}}\tilde{\mathbf{4}}$	$\mathbf{1}\tilde{\mathbf{4}}$	$\mathbf{2}\tilde{\mathbf{4}}$	$\mathbf{3}\tilde{\mathbf{4}}$	$\mathbf{4}\tilde{\mathbf{4}}$

Slate 2. Braille cells of $\mathcal{B}$ according to the elements of $\mathcal{A}$.

3. Symmetry

We consider the Braille subsets of $\mathcal{B}$ in the Slate 2

$$\{\tilde{\mathbf{4}}\mu\},\{\mathbf{4}\mu\}\quad\{\tilde{\mathbf{3}}\mu\},\{\mathbf{3}\mu\}\quad\{\tilde{\mathbf{2}}\mu\},\{\mathbf{2}\mu\}\quad\{\tilde{\mathbf{1}}\mu\},\{\mathbf{1}\mu\}. \tag{3.17}$$

with μ or $\tilde{\mu}\in\mathcal{A}$. We can write applying the swap function

$$\widetilde{(\tilde{\mathbf{4}})}\mu=\mathbf{4}\mu\quad\widetilde{(\tilde{\mathbf{3}})}\mu=\mathbf{3}\mu\quad\widetilde{(\tilde{\mathbf{2}})}\mu=\mathbf{2}\mu\quad\widetilde{(\tilde{\mathbf{1}})}\mu=\mathbf{1}\mu. \tag{3.18}$$

Hence it is convenient to introduce a mapping R from $\mathcal{B}$ into $\mathcal{B}$

$$x=\alpha\mu\mapsto R(x)=\tilde{\alpha}\mu. \tag{3.19}$$

Then we have

$$\mathcal{B} = \mathcal{B}' \cup R(\mathcal{B}') \tag{3.20}$$

in which

$$\mathcal{B}' \tag{3.21}$$

is the subset of cells of the Slate 3 and

$$R(\mathcal{B}') = \{R(x) : x \in \mathcal{B}'\} \tag{3.22}$$

is the (direct) image $R(\mathcal{B}')$ of $\mathcal{B}'$ under R [1].

1 4	**2 4**	**3 4**	**4 4**
1 3	**2 3**	**3 3**	**4 3**
1 2	**2 2**	**3 2**	**4 2**
1 1	**2 1**	**3 1**	**4 1**
$\mathbf{1\,\tilde{1}}$	$\mathbf{2\,\tilde{1}}$	$\mathbf{3\,\tilde{1}}$	$\mathbf{4\,\tilde{1}}$
$\mathbf{1\,\tilde{2}}$	$\mathbf{2\,\tilde{2}}$	$\mathbf{3\,\tilde{2}}$	$\mathbf{4\,\tilde{2}}$
$\mathbf{1\,\tilde{3}}$	$\mathbf{2\,\tilde{3}}$	$\mathbf{3\,\tilde{3}}$	$\mathbf{4\,\tilde{3}}$
$\mathbf{1\,\tilde{4}}$	$\mathbf{2\,\tilde{4}}$	$\mathbf{3\,\tilde{4}}$	$\mathbf{4\,\tilde{4}}$

Slate 3. 32 Braille cells of $\mathcal{B}'$.

Let us consider the cells of $\mathcal{B}'$ $\{\alpha\mathbf{4}\},\{\alpha\tilde{\mathbf{4}}\};\{\alpha\mathbf{3}\},\{\alpha\tilde{\mathbf{3}}\};\{\alpha\mathbf{2}\},\{\alpha\tilde{\mathbf{2}}\};\{\alpha\mathbf{1}\},\{\alpha\tilde{\mathbf{1}}\}$ in the Slate 3 with $\alpha = 1,2,3,4$. We can write

$$\alpha\mathbf{4} = \alpha\left(\widetilde{\tilde{\mathbf{4}}}\right) \quad \alpha\mathbf{3} = \alpha\left(\widetilde{\tilde{\mathbf{3}}}\right) \quad \alpha\mathbf{2} = \alpha\left(\widetilde{\tilde{\mathbf{2}}}\right) \quad \alpha\mathbf{1} = \alpha\left(\widetilde{\tilde{\mathbf{1}}}\right). \tag{3.23}$$

Hence it is convenient to introduce a mapping R from $\mathcal{B}$ into $\mathcal{B}$

$$x = \alpha\mu \quad \mapsto \quad S(x) = \alpha\tilde{\mu}. \tag{3.24}$$

According to the map results

$$\mathcal{B}' = \mathcal{B}'' \cup S(\mathcal{B}'') \tag{3.25}$$

in which

$$\mathcal{B}'' \tag{3.26}$$

is the set of cells of the Slate 4.

1 4	**2 4**	**3 4**	**4 4**
1 3	**2 3**	**3 3**	**4 3**
1 2	**2 2**	**3 2**	**4 2**
1 1	**2 1**	**3 1**	**4 1**

Slate 4. 16 Braille cells of $\mathcal{B}''$.

Can $\mathcal{B}''$ be reduced further? Yes, for example (see Slate 4), what's the difference between $\begin{vmatrix} + & + \\ + & + \\ 0 & + \end{vmatrix} = \alpha\beta$ of $\mathcal{B}''$ and $\begin{vmatrix} + & + \\ + & + \\ + & 0 \end{vmatrix} = \alpha'\beta' \in \mathcal{B}''$? How they can be related? That is $\alpha' = \beta$, $\beta' = \alpha$. This enables us to define the following function from $\mathcal{B}$ into $\mathcal{B}$

$$x = \alpha\mu \quad \mapsto \quad T(x) = \mu\alpha\,. \tag{3.27}$$

Via map T, we obtain

$$\mathcal{B}'' = \mathcal{B}''' \cup T(\mathcal{B}''')\,. \tag{3.28}$$

Let

$$\mathcal{B}''' \tag{3.29}$$

have the 10 cells of the Slate 5.

			44
		33	**43**
	22	**32**	**42**
11	**21**	**31**	**41**

Slate 5. 10 Braille cells of $\mathcal{B}'''$.

Replacing $\mathcal{B}'' = \mathcal{B}''' \cup T(\mathcal{B}''')$, $\mathcal{B}' = \mathcal{B}'' \cup S(\mathcal{B}'')$, $\mathcal{B} = \mathcal{B}' \cup R(\mathcal{B}')$ into a single one, we conclude in the next expression

$$\mathcal{B} = \left(\mathcal{B}''' \cup T(\mathcal{B}''') \cup S\left(\mathcal{B}''' \cup T(\mathcal{B}''')\right)\right) \cup R\left(\mathcal{B}''' \cup T(\mathcal{B}''') \cup S\left(\mathcal{B}''' \cup T(\mathcal{B}''')\right)\right). \tag{3.30}$$

But, taken into account that any function is distributive respect to arbitrary union of subsets of its domain [2], we have

$$\begin{aligned}\mathcal{B} = \mathcal{B}''' \cup R(\mathcal{B}''') \cup S(\mathcal{B}''') \cup T(\mathcal{B}''') \cup R \circ S(\mathcal{B}''') \cup S \circ T(\mathcal{B}''') \\ \cup R \circ T(\mathcal{B}''') \cup R \circ S \circ T(\mathcal{B}''')\end{aligned} \tag{3.31}$$

Therefore, we can reconstruct all 64 cells of the Braille code $\mathcal{B}$ from the subset $\mathcal{B}'''$ with a smaller number of elements (10 cells), R, S and T according to the last equation.

Before continuing, we have to mentioned the identity function I in $\mathcal{B}$ yet

$$x = \alpha\mu \quad \mapsto \quad I(x) = \alpha\mu\,. \tag{3.32}$$

Is there anything else to say about the functions? Of course! What happens if we compose two functions? It is easy to check with the maps R, S, and T that their compositions give us

$$R \circ R(x) = S \circ S(x) = T \circ T(x) = I \circ I(x) = I(x) = x\,. \tag{3.33}$$

Applying these properties, we can conclude that

$$R, S \text{ and } T \text{ are one-to one functions from } \mathcal{B} \text{ onto } \mathcal{B}\,. \tag{3.34}$$

They are permutations of Braille code. Let $g \in \{I, R, S, T\}$, assume $g(x) = g(y)$ then we have $g \circ g(x) = x = y = g \circ g(y)$ then f is injective, or one-to-one. Now suppose $y \in \mathcal{B}$ then $y = g \circ g(y) = g(g(y)) = g(x)$. Hence g is surjective, or onto. In other words, they are bijective or permutations of Braille code. Any of these functions verifies

$$g(\mathcal{B}) = \mathcal{B}\,. \tag{3.35}$$

That means symmetry. What about the commutative property between them?

$$\begin{aligned} R \circ T(x) &= \bar{\mu}\alpha \neq \mu\bar{\alpha} = T \circ R(x) \\ R \circ S(x) &= \bar{\alpha}\bar{\mu} = S \circ R(x) \\ T \circ S(x) &= \bar{\mu}\alpha \neq \mu\bar{\alpha} = S \circ T(x) \end{aligned} \tag{3.36}$$

We observe that the order in which these transformations are performed matters (see the **Figure 1**)

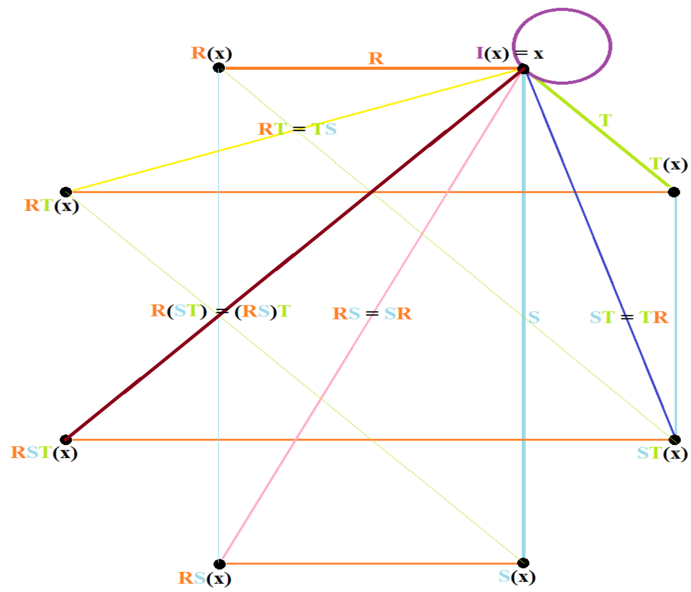


Figure 1. Diagrams associated to a cell x to illustrate associative compositions and non-commutativity.

In **Figure 1**, we represent the functions in colors R (orange) S (blue), T (green) and I (violet). Too, we have changed the symbol of composition, for example, $TR = T \circ R$. If f, g, h are elements of the set $\{I, R, S, T\}$, are they associative? Functions composition is always associative. That means the following holds true

$$f \circ (g \circ h)(x) = (f \circ g) \circ h(x). \tag{3.37}$$

4. Graph

From **Figure 1**, we deduce the next Figure.

Let's introduce what we call the (simple) graph [3] (**Figure 2**)

$$H = (V(H), E(H), \psi_H) \tag{4.38}$$

associated with a cell

$$x = \alpha\mu. \tag{4.39}$$

Symbol

$$V(H) = \left\{ \begin{array}{c} x = \alpha\mu, \\ R(x) = \bar{\alpha}\mu, \quad S(x) = \alpha\bar{\mu}, \quad T(x) = \mu\alpha, \\ RS(x) = \bar{\alpha}\bar{\mu}, \quad ST(x) = \mu\bar{\alpha}, \quad RT(x) = \bar{\mu}\alpha, \\ RST(x) = \bar{\mu}\bar{\alpha} \end{array} \right\} \tag{4.40}$$

denotes the vertex set of H, and its elements are called the vertices (or the nodes) of H.

The set

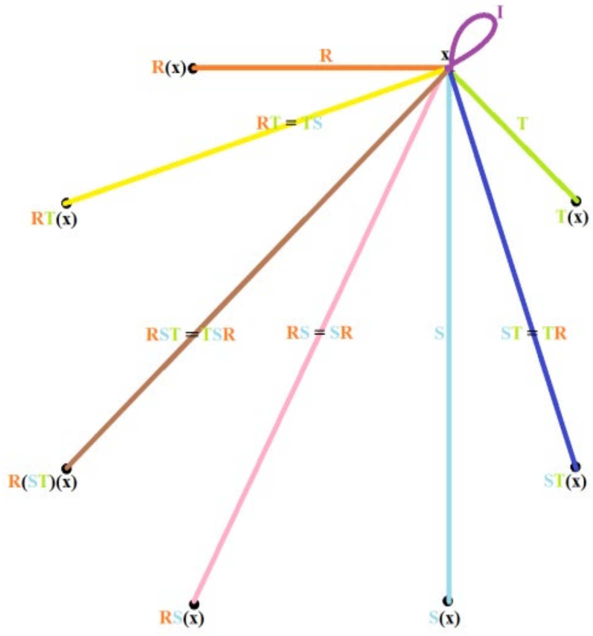


Figure 2. Graph H "Octopus", 2^3 (or "quatropus" (quadruped), 2^2) associated to a cell x.

$$E(H)=\left\{\begin{array}{c} I, \\ R,\quad S,\quad T, \\ RS,\quad ST,\quad RT, \\ RST \end{array}\right\} \tag{4.41}$$

represents the edge set of G in which its elements are the edges of H. What's ψ_H? It is an incidence function defined in the follow way

$$\psi_H(E(H))=\left\{\begin{array}{c} \psi_H(I)=\{x,I(x)\}, \\ \psi_H(R)=\{x,R(x)\},\quad \psi_H(S)=\{x,S(x)\},\quad \psi_H(T)=\{x,T(x)\}, \\ \psi_H(RS)=\{x,RS(x)\},\quad \psi_H(ST)=\{x,ST(x)\},\quad \psi_H(RT)=\{x,RT(x)\}, \\ \psi_H(RST)=\{x,RST(x)\} \end{array}\right\} \tag{4.42}$$

Observe the four Braille cells such that

$$x=x_0=\alpha\alpha. \tag{4.43}$$

For these cells we have using Figure 1's diagram the cells $y_{i=1,2,3,4}$ of the Braille set

$$\begin{array}{c} x_0=\alpha\alpha=y_1 \\ R(x_0)=\bar{\alpha}\alpha=y_2\quad S(x_0)=\alpha\bar{\alpha}=y_3\quad T(x_0)=y_1 \\ S\circ R(x_0)=\bar{\alpha}\bar{\alpha}=y_4\quad T\circ R(x_0)=y_3\quad S\circ T(x_0)=y_3 \\ R\circ S\circ T(x_0)=y_4 \end{array} \tag{4.44}$$

That implies

$$\alpha\alpha \quad \mapsto \quad \{\alpha\alpha, \bar{\alpha}\alpha, \alpha\bar{\alpha}, \bar{\alpha}\bar{\alpha}\}. \tag{4.45}$$

Remember

$$\mathcal{B}''' = \underbrace{\left\{ \begin{bmatrix} + & + \\ + & + \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} + & + \\ 0 & 0 \\ + & + \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ + & + \\ + & + \end{bmatrix}, \begin{bmatrix} + & + \\ + & + \\ + & + \end{bmatrix} \right\}}_{4}$$
$$\cup \underbrace{\left\{ \begin{bmatrix} + & + \\ 0 & + \\ + & 0 \end{bmatrix}, \begin{bmatrix} 0 & + \\ + & + \\ + & 0 \end{bmatrix}, \begin{bmatrix} + & + \\ + & + \\ + & 0 \end{bmatrix}, \begin{bmatrix} 0 & + \\ + & 0 \\ + & + \end{bmatrix}, \begin{bmatrix} + & + \\ + & 0 \\ + & + \end{bmatrix}, \begin{bmatrix} + & 0 \\ + & + \\ + & + \end{bmatrix} \right\}}_{6} \tag{4.46}$$

Inside $\mathcal{B}'''$ we find 4 cells of the form $\alpha\alpha$ and 6 of the other type $\alpha\mu$, $\mu \neq \alpha$, then the number of cells generated by $\mathcal{B}'''$ is

$$4 \times 2^2 + 6 \times 2^3 = 64. \tag{4.47}$$

5. Group

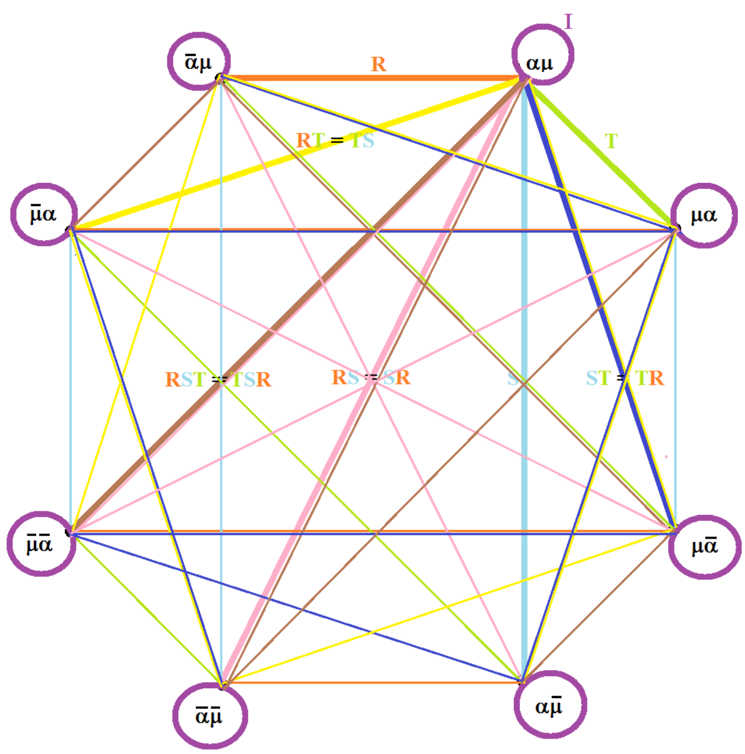


Figure 3. Group $G = (E(H), \circ)$ for $V(H)$.

Is there a group inside a Braille code?

$$G = (E(H), \circ) \tag{5.48}$$

is group. Taking into account **Figure 3**

$$E(H)=\{I,R,S,T,RS=SR,ST=TR,RT=TS,RST=TSR\}$$
$$V(H)=\left\{\begin{matrix} & \alpha\mu, & \\ \bar{\alpha}\mu, & \alpha\bar{\mu}, & \mu\alpha, \\ \bar{\alpha}\bar{\mu}, & \mu\bar{\alpha}, & \bar{\mu}\alpha, \\ & \bar{\mu}\bar{\alpha} & \end{matrix}\right\} \tag{5.49}$$

Now, consider the group's definition in the following manner. First, our first purpose is to proof that the symmetry's results

$$E(H)=\left\{\begin{matrix} & I, & \\ R, & S, & T, \\ RS, & ST, & RT, \\ & RST & \end{matrix}\right\}=\{f:f(V(H))=V(H)\}. \tag{5.50}$$

The proof consider that

$$\begin{gathered} I(V(H))=\left\{\begin{matrix} & \alpha\mu, & \\ \bar{\alpha}\mu, & \alpha\bar{\mu}, & \mu\alpha, \\ \bar{\alpha}\bar{\mu}, & \mu\bar{\alpha}, & \bar{\mu}\alpha, \\ & \bar{\mu}\bar{\alpha} & \end{matrix}\right\}=V(H) \quad R(V(H))=\left\{\begin{matrix} & \bar{\alpha}\mu, & \\ \alpha\mu, & \bar{\alpha}\bar{\mu}, & \bar{\mu}\alpha, \\ \alpha\bar{\mu}, & \bar{\mu}\bar{\alpha}, & \mu\alpha, \\ & \mu\bar{\alpha} & \end{matrix}\right\}=V(H) \\ S(V(H))=\left\{\begin{matrix} & \alpha\bar{\mu}, & \\ \bar{\alpha}\bar{\mu}, & \alpha\mu, & \mu\bar{\alpha}, \\ \bar{\alpha}\mu, & \mu\alpha, & \bar{\mu}\bar{\alpha}, \\ & \bar{\mu}\alpha & \end{matrix}\right\}=V(H) \quad T(V(H))=\left\{\begin{matrix} & \mu\alpha, & \\ \mu\bar{\alpha}, & \bar{\mu}\alpha, & \alpha\mu, \\ \bar{\mu}\bar{\alpha}, & \bar{\alpha}\mu, & \alpha\bar{\mu}, \\ & \bar{\alpha}\bar{\mu} & \end{matrix}\right\}=V(H) \end{gathered} \tag{5.51}$$

Hence, for the others functions of $E(H)$

$$\begin{gathered} RS(V(H))=R(V(H))=V(H) \quad ST(V(H))=S(V(H))=V(H) \\ RT(V(H))=R(V(H))=V(H) \quad RST(V(H))=R(V(H))=V(H) \end{gathered} \tag{5.52}$$

We have proven that all maps of the edge set are symmetries when they are applied to any point of the vertex set. Using the same technique as before to deduce that if $f,g\in E(H)$ the composition of any two transformations $f\circ g$ is an edge of the graph H, that's

$$f,g\in E(H)\Rightarrow f\circ g\in E(H). \tag{553}$$

It is easy to see that

$$f\circ g(V(H))=f(V(H))=V(H). \tag{5.54}$$

The composition is a binary operation. Associativity occurs too,

$$f,g,h\in E(H)\Rightarrow f\circ(g\circ h)=(f\circ g)\circ h. \tag{5.55}$$

There exists an element of edges $I\in E(H)$ (called the identity element) such that

$$f\circ I=f=I\circ f \tag{5.56}$$

for all $f\in E(H)$. For every $f\in E(H)$ there exists $g\in E(H)$ (called the inverse of f) such that

$$f \circ g = I = g \circ f . \tag{5.57}$$

Remembering

$$E(H) = \{I, R, S, T, RS = SR, ST = TR, RT = TS, RST = TSR\} \tag{5.58}$$

Previously, it was obtained $II = RR = I = SS = TT$.

$$\begin{gathered} RS.SR = RR = I = SS = SR.RS \\ ST.TS = SS = I = TT = TS.ST \\ RT.TR = RR = I = TT = TR.RT \\ RST.TSR = RSSR = RR = I = TT = TSST = TSR.RST \end{gathered} \tag{5.59}$$

But remarks that $ST \neq TS$ $RT \neq TR$ then $E(H)$ is non commutative or non-Abelian group.

6. Equivalence Relation

Is there a partition? We would like to show

$$\mathcal{B} = \bigcup_{x \in \mathcal{B}'''} [x] \tag{6.60}$$

in which

$$[x] = \{y \in \mathcal{B}; \exists f \in E(H), y = f(x)\} \tag{6.61}$$

is an equivalent class and representative

$$x \in \mathcal{B}''' = \left\{ \begin{vmatrix} + & + \\ + & + \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} + & + \\ 0 & + \\ + & 0 \end{vmatrix}, \begin{vmatrix} 0 & + \\ + & + \\ + & 0 \end{vmatrix}, \begin{vmatrix} + & + \\ + & + \\ + & 0 \end{vmatrix}, \begin{vmatrix} + & + \\ 0 & 0 \\ + & + \end{vmatrix}, \begin{vmatrix} 0 & + \\ + & 0 \\ + & + \end{vmatrix}, \begin{vmatrix} + & + \\ + & 0 \\ + & + \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ + & + \\ + & + \end{vmatrix}, \begin{vmatrix} + & + \\ + & 0 \\ + & + \end{vmatrix}, \begin{vmatrix} + & + \\ + & + \\ + & + \end{vmatrix} \right\}. \tag{6.62}$$

To do this, it is necessary

$$x, x' \in \mathcal{B}, x' \notin [x] \Rightarrow [x'] \cap [x] = \varnothing . \tag{6.63}$$

Suppose $[x'] \cap [x] \neq \varnothing$. Let $y \in [x'] \cap [x]$, then there exists $f, g \in E(H)$ such that

$$f(x') = y = g(x) \tag{6.64}$$

but there is h such that

$$hf(x') = x' = hg(x) . \tag{6.65}$$

That's a contradiction because $fg \in E(H)$ and

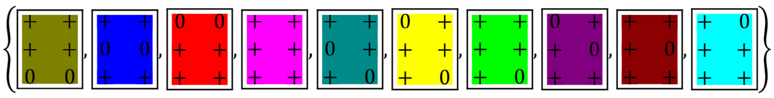
$$x' \in [x] . \tag{6.66}$$

7. An Application

In the next figure (**Slate 6**), we condensate our information on equivalence relation with a teaching-learning purpose.

We can learn mathematics playing a game "QPus". QPus is a non-zero-sum game in which all students win learning symmetry, graph, group, and equivalence relation. What materials do we use? One option is using **Slate 1**. Who are the players? Students. How do the students play QPus?

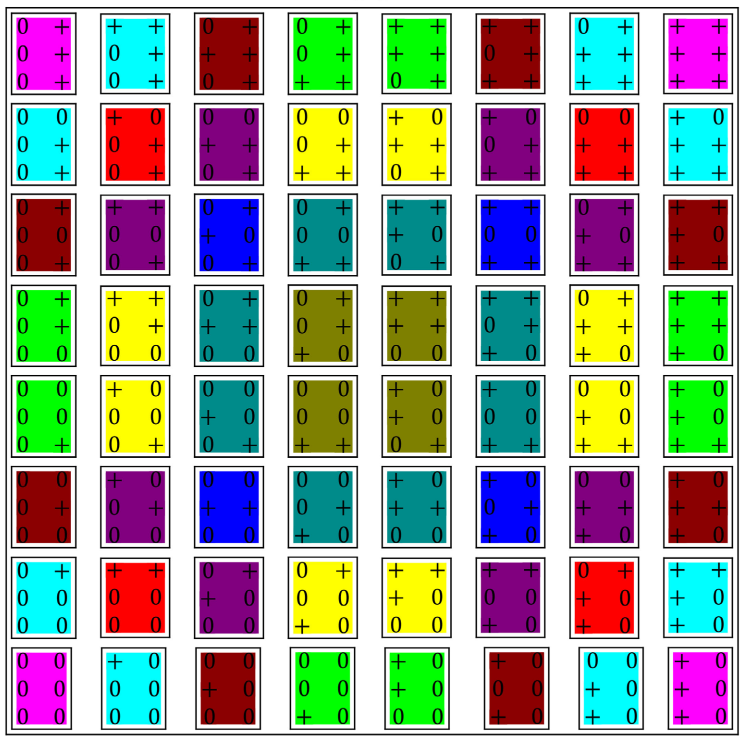
- Braille code. Slate 1.
- Equivalence relation.
  - A student chooses a Braille cell s_1 of the Slate 1.
  - All the students construct discussing the equivalence class in which s_1 is a representative according to the Equation (4.40), $[s_1] = \{\alpha\mu, \mu\alpha, \mu\bar{\alpha}, \alpha\bar{\mu}, \bar{\alpha}\bar{\mu}, \bar{\mu}\bar{\alpha}, \bar{\mu}\alpha, \bar{\alpha}\mu\}$.
  - Then students will locate the Braille cells of $[s_1]$ in the Slate 1.
  - The process is repeated whatever the students decide what will be the next Braille cell $s_{i=2,3,\cdots,10}$.
  - Game ends when the students classify every Braille cell in a quatropus or in an octopus, giving as results Slate 6.
- Symmetry, Graph and Group in a whole.

$\mathcal{B}''' =$ 

$$\alpha\alpha \in \mathcal{B}''' \quad \Longrightarrow \quad [\alpha\alpha] = \{\alpha\alpha, \alpha\bar{\alpha}, \bar{\alpha}\bar{\alpha}, \bar{\alpha}\alpha\}$$

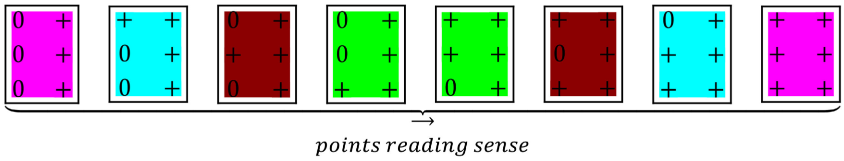
$$\alpha\mu \in \mathcal{B}''' \quad \Longrightarrow \quad [\alpha\mu] = \{\alpha\mu, \mu\alpha, \mu\bar{\alpha}, \alpha\bar{\mu}, \bar{\alpha}\bar{\mu}, \bar{\mu}\bar{\alpha}, \bar{\mu}\alpha, \bar{\alpha}\mu\}$$

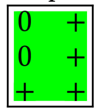
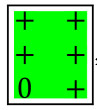
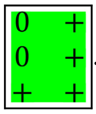
$$\mathcal{B} = \bigcup_{x \in \mathcal{B}'''} [x]$$

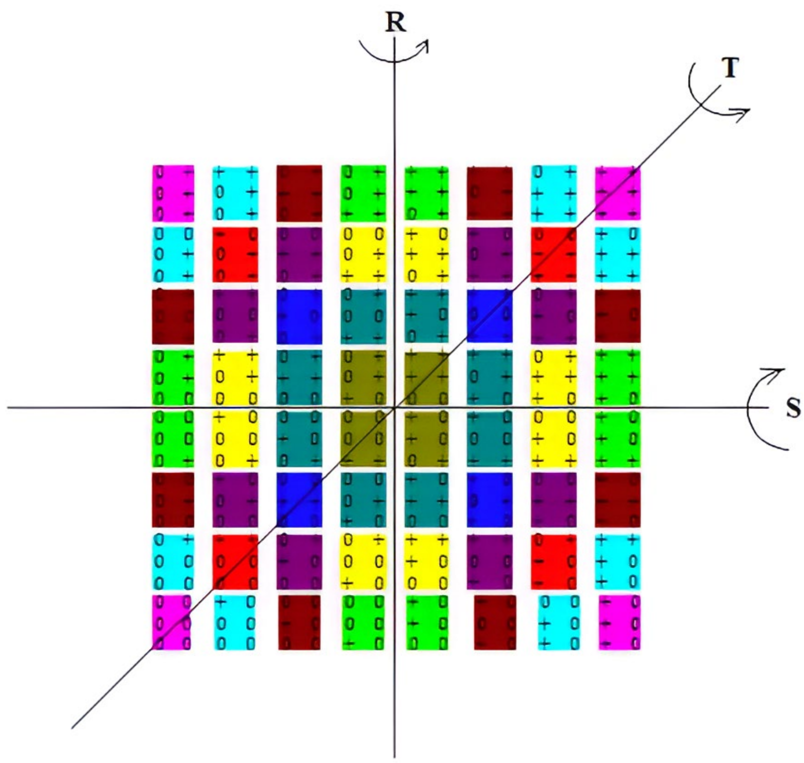


Slate 6. (Like a "square" piano). Braille code $\mathcal{B}$ partition. If to all elements of the same equivalence class we associate a specific colour, sound, musical note, Morse code, animal footprint, smell, and son on to "feel" the Braille cell, then we obtain Slate 1 "coloured".

Working with elements of Slate 7 in which every Braille cell is a point on the flat surface, we can "feel" the following facts. "Feeling" the students the first "horizontal line" until



we feel that the distance between points is a constant, which may be taken as unit to measure distance. Exploring the points given above we feel that the sound not change for first time between  and , and it begin to repeat in the order given before. In other words, something special happens when crossing . I we do the process in others horizontal lines we will find a similar phenomenon. So we can identify the "vertical line R". There are others two lines "S" (horizontal) and "T".



Slate 7. Bilateral symmetry [4] [5]. Braille set has three plane of symmetry "R", "S", and "T". Graphs and Groups may be deduced from here.

8. Conclusion

The content of this paper may be converted into tactile, audio, or haptic formats, enabling comprehension and interaction through touch and sound rather than sight. Therefore, this work is a mathematical teaching-learning resource that enables blind students to read and write on elementary concepts of naive set theory, symmetry, graphs, groups, and equivalence relations. Equations (2.12), (2.16), (3.31), and the partition of the Braille set (6.60) are the same sets using the 64 Braille cells of Slate 1 in different forms. The Braille set is symmetrical under the induced functions associated to (3.19), (3.24), (3.27), and (3.32) because satisfy (3.35). Figure 1 illustrates through a diagram about commutativity, non-commutativity and associativity properties in (3.33), (3.36), and (3.37). Figure 2 shows the graph (4.38) like an octopus or quatropus and the group (5.48) appears in Figure 3. In this way, this paper demonstrates that mathematics is capable of uniting many things by summarizing or showing immersed objects that can be discovered, motivating the curiosity in students' and teachers' minds.

This work also includes questions such as: what are the consequences if we change set (29)? Are there always 4 quadrupeds and 6 octopuses? Yes, but why?

- Any Braille cell belongs to a quatropus or an octopus. We know that the Braille set $\mathcal{B}$ is a set of 64 elements of the form $x=\alpha\mu$. If $x=\alpha\alpha$ (or $\mu=\alpha$) then we have a quatropus $[\alpha\alpha]=\{\alpha\alpha,\alpha\bar{\alpha},\bar{\alpha}\bar{\alpha},\bar{\alpha}\alpha\}$. If $x=\alpha\mu$ (or $\mu\neq\alpha$) then we have an octopus $[\alpha\mu]=\{\alpha\mu,\mu\alpha,\mu\bar{\alpha},\alpha\bar{\mu},\bar{\alpha}\bar{\mu},\bar{\mu}\bar{\alpha},\bar{\mu}\alpha,\bar{\alpha}\mu\}$. We have proven that any Braille cell belongs to a quatropus or an octopus.
- The Braille set contains 4 quatropus. If we choose an element $\alpha\in\mathcal{R}$, $[\alpha\alpha]=\{\alpha\alpha,\alpha\bar{\alpha},\bar{\alpha}\bar{\alpha},\bar{\alpha}\alpha\}=[\bar{\alpha}\bar{\alpha}]$. Then to the pair of elements α and $\bar{\alpha}$ of $\mathcal{R}$ corresponds the first octopus $[\alpha\alpha]$. Working in the set $\mathcal{R}\setminus\{\alpha,\bar{\alpha}\}=\{\rho\in\mathcal{R};\rho\neq\alpha,\rho\neq\bar{\alpha}\}$. Let $\beta\in\mathcal{R}$ such that $\beta\neq\alpha$ and $\beta\neq\bar{\alpha}$ then $[\beta\beta]=\{\beta\beta,\beta\bar{\beta},\bar{\beta}\bar{\beta},\bar{\beta}\beta\}$ and $[\alpha\alpha]$ are disjoints sets. Let the element $\gamma\in\mathcal{R}\setminus\{\alpha,\bar{\alpha},\beta,\bar{\beta}\}$ then we have three disjoints sets: $[\alpha\alpha]$, $[\beta\beta]$, and $[\gamma\gamma]$. Finally, if $\delta\in\mathcal{R}\setminus\{\alpha,\bar{\alpha},\beta,\bar{\beta},\gamma,\bar{\gamma}\}$ we obtain our last quatropus $[\delta\delta]$, and $\mathcal{R}\setminus\{\alpha,\bar{\alpha},\beta,\bar{\beta},\gamma,\bar{\gamma},\delta,\bar{\delta}\}=\varnothing$. Let $\varepsilon_1=\alpha$, $\varepsilon_2=\beta$, $\varepsilon_3=\gamma$, and $\varepsilon_4=\delta$.
- The Braille set contains 6 octopuses. The set of all octopus it will be the set given by $\mathcal{T}=\mathcal{B}\setminus([\alpha\alpha]\cup[\beta\beta]\cup[\gamma\gamma]\cup[\delta\delta])$, which have $2^6-4\times2^2=48$. The number of octopus it will be $\frac{48}{8}=6$. Let $\varepsilon_{i=5,6,\cdots,10}\in\mathcal{T}$ such that $\varepsilon_5\in\mathcal{T}$, and for the other $\varepsilon_{j=6,7,\cdots,10}\in\mathcal{T}\setminus\left(\bigcup_{k=5}^{j-1}[\varepsilon_k]\right)$. However, the sets $[\varepsilon_{k=5,6,\cdots,10}]$ are pairwise disjoint sets, and $\mathcal{T}\setminus\left(\bigcup_{k=5}^{10}[\varepsilon_k]\right)=\varnothing$.
- If we define the set $\mathcal{B}^*=\{\varepsilon_{k=1,2,\cdots,10}\}$ we have probed that $\mathcal{B}=\bigcup_{x\in\mathcal{B}^*}[x]$ and $2^6=4\times2^2+6\times2^3$.

QPus may be modify to motivate students-teacher's curiosity using others representatives or surfaces on 3D objets.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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