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Solution of Some Boundary Problems of Stationary Oscillation of Moment Theory of Elasticity Using Fourier Integral Transformation

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Abstract

We have considered the first version of the moment theory of elasticity, where $\Omega = \frac{1}{2} rotU$, where U is a displacement vector, and Ω-rotation vector. Using Fourier integral transformation, some boundary problems of the moment theory of elasticity, in particular, stationary oscillation problems for a semiplane and an endless strip, have been effectively solved.

Keywords

Fourier Integral Transformation, Displacement Vector, Rotation Vector, Semiplane, Endless Strip

1. Introduction

In the classic elasticity theory, a solid body is considered as a three-dimensional Euclidean space, points of which coincide with body particles [1] [2]. Interaction between parts of this solid body model occurs through central forces only; in particular, a surface element dA load is transmitted by the main PdA vector only. This assumption leads us to a symmetric, stressed, and deformed condition. Classic elasticity theory very well explains events taking place in bodies under pressure in all cases, when, let's say, a "granular" structure of a real body under consideration doesn't count as an essential characteristic of this event. Though the classic elasticity theory doesn't satisfactorily explain some events in those cases, the material microstructure plays an essential role.

Even Voigt W. has attempted to correct this deficiency of the classic elasticity theory using the following additional assumption that a surface element load is transmitted not only by PdA vector, but by principal $\mathrm{d}A$ moment—this assumption leads us to the fact that not only σ_{ij} stress, but couple-stress μ_{ij} acts on a volume element. In general, this new theory was developed by the Cosserat brothers (Cosserat E. and Cosserat F.). While classic elasticity theory considers a material particle as a point and describes the deformed state by means of point positions (coordinates), in contradistinction, the Cosserat brothers consider each body particle as a small solid body. Deformation of such a body is described not only by the displacement vector, but also by the rotation vector. The mentioned theory, which considers a new model of an elastic body, is called the moment theory of elasticity. There is no general consensus in the moment theory of elasticity when describing the displacement of a solid body particle. The concept of classic elasticity theory is kept unchanged in the works of [3] Mindlin R., Kuvshinsky E., and Aero E., who assume that displacement and small rigid rotation are completely described, if we take that $\Omega = \frac{1}{2} rotU$. This version of the moment theory of elasticity is called a first version (variant). [1] Nowacki W., Palmov N. *et al.*, alongside a similar displacement field of particles of the body under consideration, introduce a vector field, kinematically dependent on it, which characterises small rotations of body particles—this version is called the second version (variant) in the moment theory of elasticity.

2. Methods

We consider the first version of the moment theory of elasticity. The basic equations of equilibrium in the moment theory of elasticity in terms of displacement components are as follows:

$$\Delta U + \frac{1}{1-2\nu} graddivU + l^2 \Delta rot\Omega = 0$$

In case of stationary oscillation, we will have

$$\Delta U + \frac{1}{1-2\nu} graddivU + l^2 \Delta rot\Omega + k^2 U = 0, \tag{1}$$

where ν is Poisson ratio, l is new elasticity constant (when $l = 0$, we obtain equations of the classic elasticity theory), U is a displacement vector, and $\Omega = \frac{1}{2} rotU$ is rotation vector. Let us write the vector equality (1) in scalar form:

$$\begin{gathered} \Delta u + \frac{1}{1-2\nu}\frac{\partial\theta}{\partial x} + l^2 \frac{\partial \Delta\omega}{\partial y} + k^2 u = 0 \\ \Delta v + \frac{1}{1-2\nu}\frac{\partial\theta}{\partial y} - l^2 \frac{\partial \Delta\omega}{\partial x} + k^2 v = 0 \end{gathered} \tag{2}$$

where

$$\theta = divU\left(u,v\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \omega = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{3}$$

Let's consider the problem: assume that $y = 0$ at the boundary of endless $y > 0$ semiplane. There is given $u(x,0) = u_0$, $v(x,0) = v_0$, $\omega(x,0) = \omega_0$, where u_0, v_0, $\omega_0 \in L(-\infty;+\infty)$. Let us determine the displacement damping at infinity.

Let us note that $L \equiv L(R)$ is a designation of a class of function integrated over R with a norm $\|f\| = \int |f(x)| \mathrm{d}x$, $\hat{\varphi}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(x) \mathrm{e}^{-itx} \mathrm{d}x \ (-\infty < x < +\infty)$. $\varphi(x) \in L$ is a Fourier transformation of a function and t is a Fourier transformation parameter [4].

By applying of *div* and *rot* operations on Equation (1), we obtain

$$\Delta\theta + k^2 h\theta = 0, \ \Delta\left[\omega - k^2\omega\right] + k^2\omega = 0 \tag{4}$$

where

$$h = \frac{1-2\nu}{2(1-\nu)} \tag{5}$$

Let's rewrite (2) in the following form

$$\begin{gathered} \Delta u + k^2 u = -\frac{1}{1-2\nu}\frac{\partial\theta}{\partial x} - l^2\frac{\partial\Delta\omega}{\partial y} \\ \Delta v + k^2 v = -\frac{1}{1-2\nu}\frac{\partial\theta}{\partial y} + l^2\frac{\partial\Delta\omega}{\partial x} \end{gathered} \tag{6}$$

From Equations (5) and (6), using Fourier transformation (in terms of x variable), we obtain

$$\begin{gathered} \frac{\mathrm{d}^2\hat{\theta}}{\mathrm{d}y^2} - \left(t^2 - hk^2\right)\hat{\theta} = 0 \\ l^2\frac{\mathrm{d}^4\hat{\omega}}{\mathrm{d}y^4} - \left(1 + 2t^2l^2\right)\frac{\mathrm{d}^2\hat{\omega}}{\mathrm{d}y^2} + \left(t^2 + t^4l^2 - k^2\right)\hat{\omega} = 0 \end{gathered} \tag{7}$$

$$\begin{gathered} \frac{\mathrm{d}^2\hat{u}}{\mathrm{d}y^2} - \left(t^2 - k^2\right)\hat{u} = -\frac{it}{1-2\nu}\hat{\theta} - l^2\left[\frac{\mathrm{d}^3\hat{\omega}}{\mathrm{d}y^3} - t^2\frac{\mathrm{d}\hat{\omega}}{\mathrm{d}y}\right] \\ \frac{\mathrm{d}^2\hat{v}}{\mathrm{d}y^2} - \left(t^2 - k^2\right)\hat{v} = -\frac{1}{1-2\nu}\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}y} + l^2\left[it\frac{\mathrm{d}^2\hat{\omega}}{\mathrm{d}y^2} - it^3\hat{\omega}\right] \end{gathered} \tag{8}$$

Taking into account that we seek the solution damped at infinity, the general solution of Equation (7) will be represented as follows:

$$\hat{\theta} = A\mathrm{e}^{\lambda_1 y} \ \text{ and } \ \hat{\omega} = B\mathrm{e}^{\lambda_2 y} + C\mathrm{e}^{\lambda_3 y} \tag{9}$$

where

$$\begin{gathered} \lambda_1 = -\sqrt{t^2 - hk^2} \\ \lambda_2 = -\sqrt{\frac{1 + 2l^2t^2 + \sqrt{1 + 4l^2k^2}}{2l^2}} \\ \lambda_3 = -\sqrt{\frac{1 + 2l^2t^2 - \sqrt{1 + 4l^2k^2}}{2l^2}} \end{gathered}$$

Taking Equation (9) into account in the right-hand side of Equation (8), we obtain for its general solution:

$$
\begin{aligned}
\hat{u} &= -\frac{A_i t}{\left(\lambda_1^2 - t^2 + k^2\right)(1-2\nu)} \mathrm{e}^{\lambda_1 y} - \frac{B\lambda_2 l^2\left(\lambda_2^2 - t^2\right)}{\lambda_2^2 - t^2 + k^2} \mathrm{e}^{\lambda_2 y} \\
&\quad - \frac{C\lambda_3 l^2\left(\lambda_3^2 - t^2\right)}{\lambda_3^2 - t^2 + k^2} \mathrm{e}^{\lambda_3 y} + D\mathrm{e}^{-\sqrt{t^2-k^2}\,y} \\
\hat{v} &= -\frac{A\lambda_1}{\left(\lambda_1^2 - t^2 + k^2\right)(1-2\nu)} \mathrm{e}^{\lambda_1 y} - \frac{itBl^2\left(\lambda_2^2 - t^2\right)}{\lambda_2^2 - t^2 + k^2} \mathrm{e}^{\lambda_2 y} \\
&\quad + \frac{itCl^2\left(\lambda_3^2 - t^2\right)}{\lambda_3^2 - t^2 + k^2} \mathrm{e}^{\lambda_3 y} + E\mathrm{e}^{-\sqrt{t^2-k^2}\,y}
\end{aligned}
\tag{10}
$$

A, B, C, D, E are constants in Equation (10), which have to be considered. As far as the dependence (3) remains valid, the mentioned constants are interdependent ones. Fourier transformation of equality (3) will give us:

$$
\hat{\theta} = it\hat{u} + \frac{\mathrm{d}\hat{v}}{\mathrm{d}y}, \ \hat{\omega} = it\hat{v} - \frac{\mathrm{d}\hat{u}}{\mathrm{d}y} \tag{11}
$$

Using Equations (9)-(11), we obtain

$$
itD - E\sqrt{t^2 - k^2} = 0, \ itE + D\sqrt{t^2 - k^2} = 0 \tag{12}
$$

Since for homogeneous (12) systems $\Delta = \begin{vmatrix} it & -\sqrt{t^2-k^2} \\ \sqrt{t^2-k^2} & it \end{vmatrix} \neq 0$, it will have a trivial solution only, *i.e.* $D = E = 0$. By Fourier transformation of boundary conditions, we obtain

$$
\begin{aligned}
\hat{u}_o &= -\frac{it}{k^2 h} A + \frac{l\left(1+\sqrt{1+4l^2k^2}\right)\left(\sqrt{1+2l^2t^2+\sqrt{1+4l^2k^2}}\right)}{\sqrt{2}\left(1+2l^2t^2+\sqrt{1+4l^2k^2}\right)} B \\
&\quad + \frac{l\left(1-\sqrt{1+4l^2k^2}\right)\left(\sqrt{1+2l^2t^2-\sqrt{1+4l^2k^2}}\right)}{\sqrt{2}\left(1+2l^2t^2-\sqrt{1+4l^2k^2}\right)} C
\end{aligned}
$$

$$
\hat{v}_o = \frac{\sqrt{t^2 - hk^2}}{k^2 h} A + \frac{itl^2\left(1+\sqrt{1+4l^2k^2}\right)}{1+2l^2k^2+\sqrt{1+4l^2k^2}} B + \frac{itl^2(1+\sqrt{1-\sqrt{1+4l^2k^2}}}{1+2l^2k^2-\sqrt{1+4l^2k^2}} C
$$

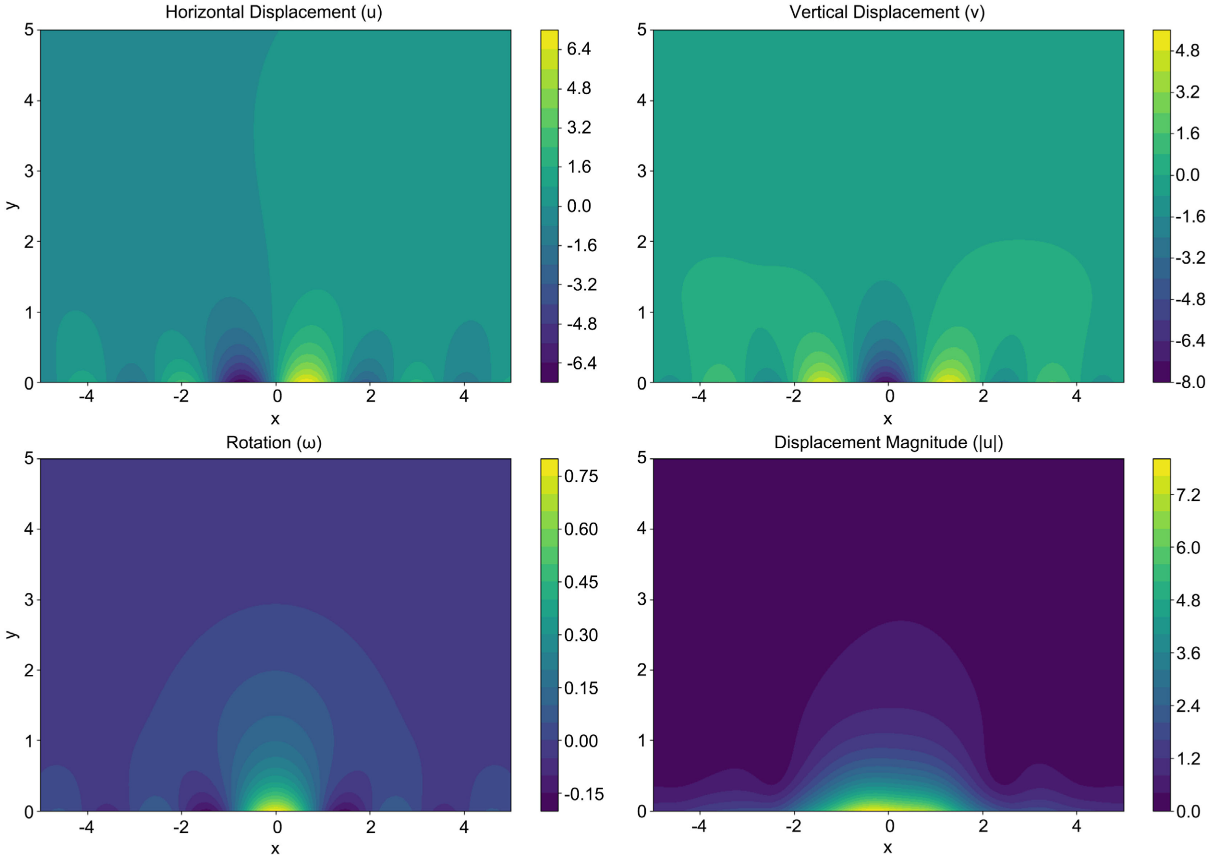
$\hat{\omega}_o = B + C$ or else, there are three conditions for three unknowns that make it possible to determine these unknowns. It should be noted as well that displacements may be expressed not only by the given function, but by their Fourier transformation, as well.

The case of infinite slab (layer) is also studied, in particular, when displacements and rotation are given $u\big|_{y=\pm h} = f_1^{\pm}$, $v\big|_{y=\pm h} = f_2^{\pm}$, $\omega\big|_{y=\pm h} = f_3^{\pm}$ on the boundary of infinite slab $|y| \le h$, $-\infty < x < +\infty$ in this case we have not a condition of damping at infinity, but boundary conditions, and taking equalities (9)-(11) into consideration, six equations for six unknowns are obtained.

3. Results and Discussions

3.1. Visualization of Displacement and Rotation Fields

In **Figure 1**, we see a visualization of solutions obtained using Fourier integral transformation. The top row shows the horizontal displacement (u) and vertical displacement (v), while the bottom row shows the rotation field (ω) and the total displacement magnitude. These fields were computed by numerically evaluating the inverse Fourier transform of the analytical solution derived in Equations (9) and (10).


Horizontal Displacement (u)

Vertical Displacement (v)

Rotation (ω)

Displacement Magnitude (|u|)

Figure 1. Visualization of solutions obtained using Fourier integral transformation.

3.2. Effect of Couple Stress Parameter

Figure 2 demonstrates the effect of the couple stress parameter (l) on displacement magnitude. As l increases, the wave characteristics and penetration depth change noticeably, demonstrating the influence of couple stresses on the solution behavior.

3.3. Comparative Analysis with Classical Elasticity Theory

Comparative profiles showing how the couple stress parameter affects horizontal displacement and rotation at a fixed depth ($y = 0.5$) are depicted in **Figure 3**. The classic elasticity theory ($l \approx 0$) shows different behavior compared to the moment elasticity theory with various l values. To further illustrate the differences between classical and moment elasticity theories.

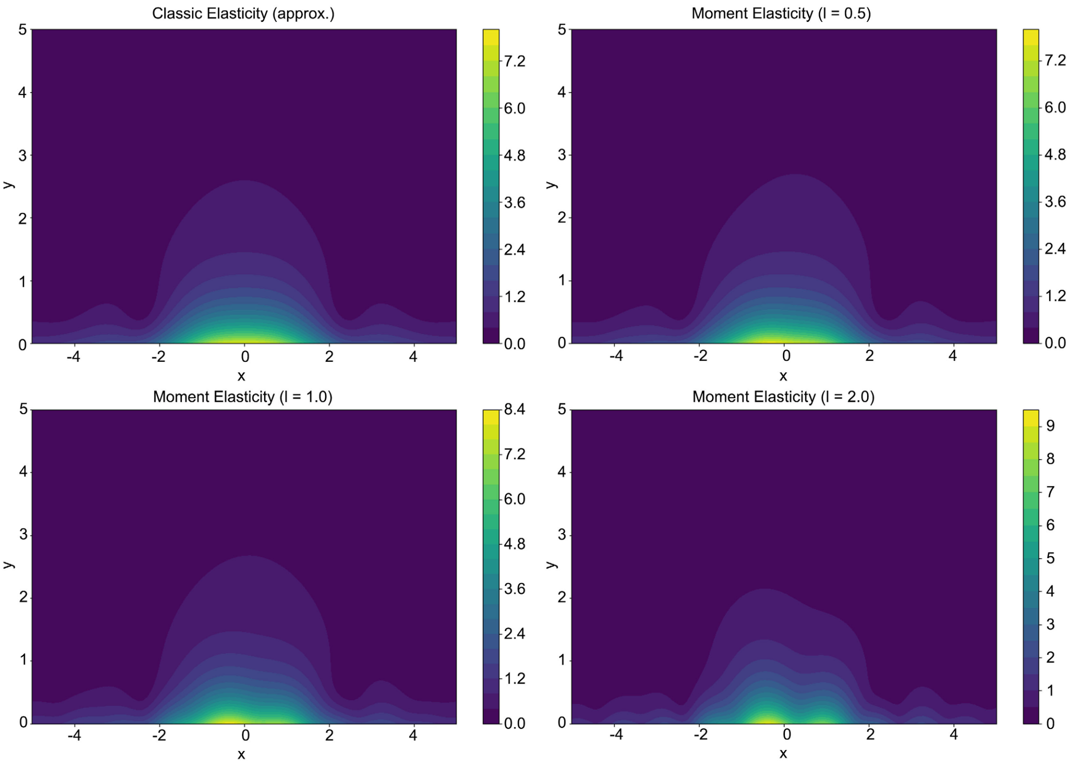


Figure 2. Effect of the couple stress parameter (l) on displacement magnitude.

Figure 3 presents comparative displacement and rotation profiles at a fixed depth.

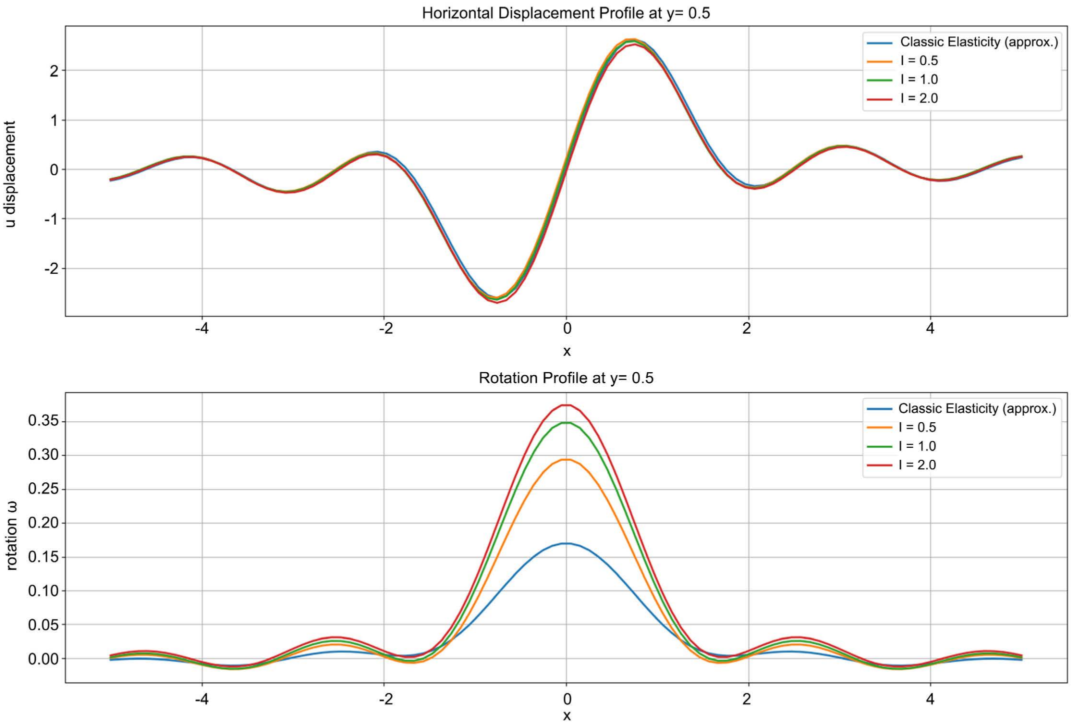


Figure 3. Couple stress parameter effects on horizontal displacement and rotation at a fixed depth.

3.4. Validation of the Solution

A mathematical validation of the solution is shown in **Figure 4**, showing the exponential decay with depth predicted by the theory. The numerical solution closely follows the theoretical decay rate determined by the eigenvalues λ_1, λ_2, λ_3, confirming the accuracy of the Fourier integral transformation approach.

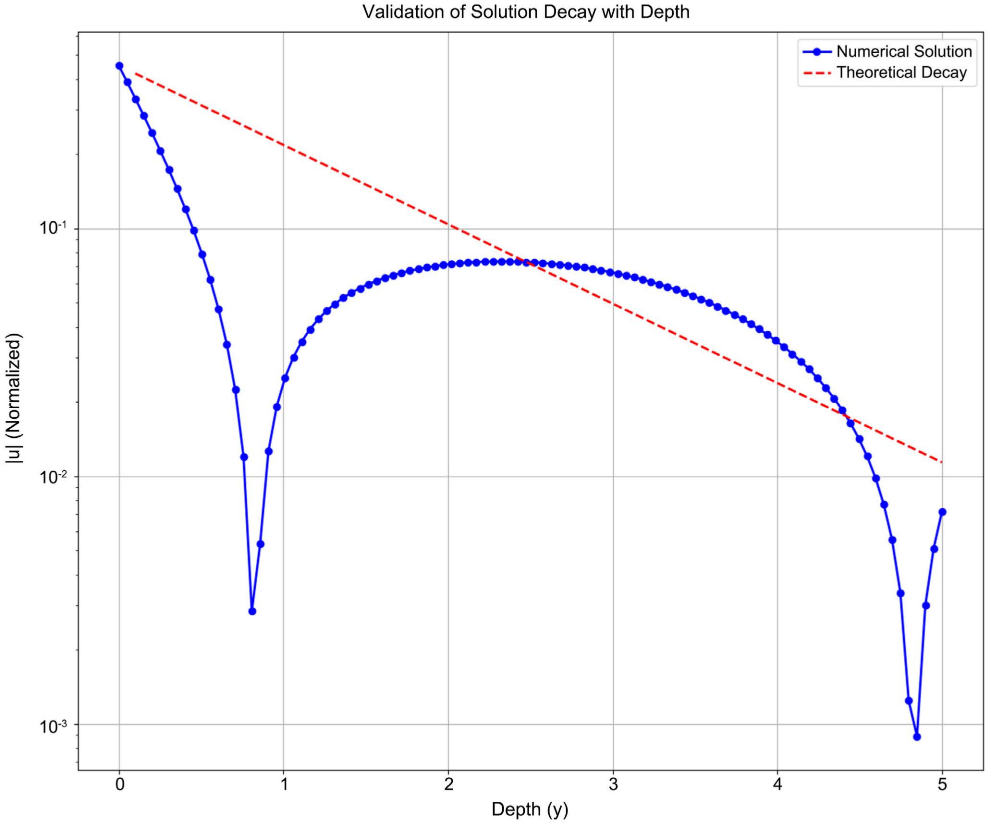


Figure 4. Exponential decay with depth.

3.5. Computational Implementation

The numerical visualizations presented in this section were generated using Python programming language with the following libraries: NumPy for numerical computations, Matplotlib for graphical plotting, and SciPy for scientific computing functions. The inverse Fourier transforms were computed using numerical integration methods, with appropriate discretization of the Fourier parameter space to ensure convergence of the solutions.

4. Conclusion

In conclusion, it should be noted that the method based on Fourier integral transformation is considered effective when solving a wide range of problems of classic elasticity theory and the moment theory of elasticity. The numerical visualizations presented demonstrate the physical significance of the couple stress parameter and validate the mathematical approach. The study discussed in the work confirms this suggestion.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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