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# $L^\infty$-Asymptotic Behavior of the Variational Inequality Related to American Options Problem

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## Abstract

**We study the approximation of variational inequality related to American options problem. A simple proof to asymptotic behavior is also given using the theta time scheme combined with a finite element spatial approximation in uniform norm, which enables us to locate free boundary in practice.**

## Keywords

**American Options, Finite Elements, Parabolic Variational Inequalities, Fixed Point, Asymptotic Behavior**

## 1. Introduction

Since the work of Black-Scholes in 1973 see [1], the financial markets have expanded considerably and traded products are increasingly numerous and sophisticated. Most widespread of these products are the options. The basic options are the options to sell and purchase, respectively called put and call. If option can be exercised at any time until maturity, we speak about American option otherwise it is a European option.

The two researchers provide a method of evaluation of European options by solving a partial derivative equation called (black-Scholes's equation). However, we cannot get explicit formula for pricing of American options, even the most simple. The formalization of the problem of pricing American options as variational inequality, and its discretization by numerical methods, appeared only rather tardily in the article of Jaillet, Lamberton and Lapeyre see [2]. A little later, the book of Wilmott, Dewynne and Howison see [3] has made it much more accessible the pricing by L.C.P from American Option problem. For the problems at free boundary several numerical results have been obtained for parabolic and elliptic variational and quasi-variational inequality see [4]-[8].

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For our part, we are interested to asymptotic behavior of V.I related to the American options problem. Where we adapted to our problem result obtained in [4] and we eliminated an additional factor $\log h$. We discretize the space $H^1(\Omega)$ by a space $V^h \subset H^1(\Omega)$ constructed from polynomials of degree 1 and the time by $\theta$-scheme. Subsequently, we demonstrated the error estimate between the continuous solution and the discrete solution of the problem given by:

For $\theta \geq \frac{1}{2}$, we have

$$\left\|u_h^{\theta,n} - u^\infty\right\|_\infty \leq C\left[h^2\left|\log h\right|^2 + \left(\frac{1}{1+\beta\theta\Delta t}\right)^n\right],$$

and for $0 \leq \theta < \frac{1}{2}$, we have

$$\left\|u_h^{\theta,n} - u^\infty\right\|_\infty \leq C\left[h^2\left|\log h\right|^2 + \left(\frac{2Ch^2}{2Ch^2+\beta\theta(1-2\theta)}\right)^n\right].$$

We used the uniform norm, because it is a realistic norm, which gives us the approximation described above and which enables us to locate the free boundary, a crucial thing in practice of the American options.

The paper is organized as follows. In Section 2, we give the problem of American options as a parabolic variational inequality. In Section 3, we discretize by the finite elements method and we deal the stability of $\theta$-scheme for our V.I. In Section 4, we adapt to our problem results obtained for similar problems see ([7] [9]) namely a contraction associated with our problem which allows us to define an algorithm of Bensoussan-Lions [10]. Finally, in Section 5, we establish the estimate of the asymptotic behavior of $\theta$-scheme by the uniform norm for American options problem.

## 2. Formulation of American Options Problem as Variational Inequality

In this section, we recall the context of our problem (see [11]-[13]). An American option is a contract which gives the right to receive the payoff $h\left(S_t^{(i)}\right)$ at some time $t \in [0,T]$, where $T < +\infty$. This payoff is then given as a function of the prices $\left(S_t^{(i)}\right)_{i=1,\cdots,n}$ at the time $t$ of $n$ financial products constituting the underlying asset. Such as these prices are strictly positive, we set

$$X_t^{(i)} = \log\left(S_t^{(i)}\right) \quad \text{for } i = 1,\cdots,n, \tag{1}$$

and we express the payoff under the form

$$h\left(S_t^{(i)}\right) = \psi\left(X_t^{(i)}\right), \tag{2}$$

where

$$\psi : \mathbb{R}^n \to \mathbb{R} \text{ is a given regular function.} \tag{3}$$

We assume that the following stochastic differential equation is satisfied by the logarithmic transformation of the prices

$$\mathrm{d}X_t^{(i)} = \left(r - \frac{1}{2}\sum_{j=1}^n \sigma_{ij}^2\right)\mathrm{d}t + \sum_{j=1}^n \sigma_{ij}\mathrm{d}W_t^{(j)}, \quad i = 1,\cdots,n. \tag{4}$$

where $r > 0$ is the interest rate, $\left(\sigma_{ij}\right)_{i,j=1,\cdots,n}$ is invertible Volatility matrix and $\left(W_t\right)_{t\in[0,T]}$ is a standard $n$-dimensional Brownian motion defined on a probability space $(\Omega, F, P)$.

### The Continuous Problem

Under some assumptions on financial markets (no-arbitrage principle) see [2] [14] and the above assumptions one can prove, that $U(x,t)$ is a solution of the following parabolic inequality:

$$\begin{cases} \dfrac{\partial U}{\partial t}+\displaystyle\sum_{i,j=1}^{n}\left(\frac{1}{2}\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk}\right)\frac{\partial^2 U}{\partial x_i\partial x_j}+\sum_{j=1}^{n}\left(r-\frac{1}{2}\sum_{i=1}^{n}\sigma_{ij}^2\right)\frac{\partial U}{\partial x_j}-rU\le 0,\ \text{for } (x,t)\in\mathbb{R}^n\times(0,T) \\ U(x,t)\ge\psi(x)\ \text{for } (x,t)\in\mathbb{R}^n\times(0,T) \\ \left(\dfrac{\partial U}{\partial t}+\displaystyle\sum_{i,j=1}^{n}\left(\frac{1}{2}\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk}\right)\frac{\partial^2 U}{\partial x_i\partial x_j}+\sum_{j=1}^{n}\left(r-\frac{1}{2}\sum_{i=1}^{n}\sigma_{ij}^2\right)\frac{\partial U}{\partial x_j}-rU\right)\left(\psi(x)-U(x,t)\right)=0,\ \text{for } (x,t)\in\mathbb{R}^n\times(0,T) \\ U(x,T)=\psi(x)\ \text{for } x\in\mathbb{R}^n \end{cases} \tag{5}$$

Now we will give the variational inequality related to American options problem in a more compact form, where we starts by giving new notations and imposed certain conditions.

By a change of variable $u(t,x)=U(T-t,x)$, the problem (5) becomes:

Find $u\in K$ solution of

$$\frac{\partial u}{\partial t}+Au\ge f\ \text{ for } (x,t)\in\Omega\times(0,T), \tag{6}$$

where $K$ is a closed convex set defined as follows:

$$K=\left\{v(t,x)\in L^2\left(0,T;H^1(\Omega)\right),\ v(t,x)\ge\psi(x),\ v(0,x)=\psi(x)\,\text{dans }\Omega\right\}, \tag{7}$$

with

$$\psi(x)\in H^1(\Omega), \tag{8}$$

and $\Omega$ is a bounded smooth domain in $\mathbb{R}^n, n\ge 1$, with boundary $\Gamma$.

*A* is an operator defined over $H^1(\Omega)$ by:

$$Au=-\sum\nolimits_{i,j=1}^{n}a_{ij}\frac{\partial}{\partial x_j}\frac{\partial u}{\partial x_j}+\sum\nolimits_{j=1}^{n}b_j\frac{\partial u}{\partial x_j}+a_0u, \tag{9}$$

and the coefficients: $a_{ij}, b_j, a_0$, where $1\le i\le n$ and $1\le j\le n$ are satisfy the following conditions:

$$a_{i,j}=a_{j,i}=\frac{1}{2}\sum\nolimits_{k=1}^{n}\sigma_{ik}\sigma_{jk};\ \ a_0=r\ge\beta>0,\ \text{where } \beta \text{ is constant}, \tag{10}$$

$$\sum\nolimits_{i,j=1}^{n}a_{i,j}\varepsilon_i\varepsilon_j\ge\gamma|\varepsilon|^2,\ \varepsilon\in\mathbb{R}^n,\ \gamma>0,\ x\in\bar{\Omega}, \tag{11}$$

$$b_j=\left(\frac{1}{2}\sum\nolimits_{i=1}^{n}\sigma_{ij}^2-r\right). \tag{12}$$

*f* is a positive function.

For more detail on the parabolic inequality associated with American options problem (see [2] [13] [15]-[17]).

We can reformulate the problem (6) to the following parabolic variational inequality:

$$\left(\frac{\partial u}{\partial t},v-u\right)+a(u,v-u)\ge(f,v-u),\ \ v\in K, \tag{13}$$

where $a(.,.)$ is a continuous bilinear form associated with operator *A* defined in (9). Namely,

$$a(u,v)=\int_\Omega\left(\sum\nolimits_{i,j=1}^{n}a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_j}+\sum\nolimits_{j=1}^{n}b_j(x)\frac{\partial u}{\partial x_j}v+a_0(x)uv\right)\mathrm{d}x. \tag{14}$$

**Theorem 1** (Cf. [10]): If $\psi\in H^1(\Omega)$, the problem (13) has an unique solution $u\in K$ Moreover, one has

$$u\in L^2\left(0,T:H^1(\Omega)\right)\ et\ \frac{\partial u}{\partial t}\in L^2\left(0,T:L^2(\Omega)\right). \tag{15}$$

## 3. Study of the Discrete Problem

We decomposed $\Omega$ into triangles and let $\tau_h$ denotes the set of all those elements, where $h>0$ is the mesh size. We assume that the family $\tau_h$ is regular and quasi uniform. Let $V^h$ denote the standard piecewise linear finite element space, and $\mathbb{A}$ be the matrix with generic coefficients $a(\varphi_i,\varphi_j)$ where $\varphi_i$, $i=\{1,2,\cdots,m(h)\}$, are the basis function of the space $V^h$, defined by $\varphi_i(M_j)=\delta_{ij}$, where $M_j$ is a vertex of the considered triangulation.

We introduce the following discrete spaces $V^h$ of finite element constructed from polynomials of degree 1:

$$\left\{v_h \in L^2\left(0,T:H^1(\Omega)\right)\cap C\left(0,T:H^1(\Omega)\right), \text{such that } v_h\vdots_k \in P_1, k\in\tau_h\right\} \tag{16}$$

and

$$K^h=\left\{v_h\in V^h, v\geq r_h\psi, v_h(x,0)=r_h\psi \text{ in } \Omega\right\}. \tag{17}$$

We consider $r_h$ be the usual interpolation operator defined by: $\forall v_h \in L^2\left(0,T:H^1(\Omega)\right)\cap C\left(0,T:H^1(\Omega)\right)$

$$r_h v=\sum_{j=1}^{m(h)} v(M_j)\varphi_j(x) \tag{18}$$

**The discrete maximum principle assumption (d.m.p):** We assume that the matrix $\mathbb{A}$ defined above is an *M*-matrix (Cf. [18]).

Theorem 2 (Cf. [19]): Let us assume that the bilinear form $a(.,.)$ is weakly coercive in $H^1(\Omega)$, there exists two constants $\alpha>0$ and $\lambda>0$ such that

$$a(u_h,u_h)+\lambda\|u_h\|_2\geq\alpha\|u_h\|_1 \tag{19}$$

**Notation:**

$(.,.)$ denotes the inner product in $L^2(\Omega)$.

$\|.\|_{L^\infty}=\|.\|_\infty$, $\|.\|_{H^1}=\|.\|_1$ and $\|.\|_{L^2}=\|.\|_2$

### 3.1. Discretization

We discretize the space $H^1(\Omega)$ by a space discretization of finite dimensional $V^h\in H^1(\Omega)$ constructed from polynomials of degree 1 and for the regularity of the solution see [20]. In a second step, we discretize the problem with respect to time using the $\theta$-scheme. Therefore, we search a sequence of elements $u_h^n\in V^h$ which approaches $u^n(t_n), t_n=n\Delta t$, with initial data $u_h^0=u_{0h}$.

We apply the finite element method to approximate inequality (13), and the semi-discrete P.V.I takes the form of

$$\left(\frac{\partial u_h}{\partial t}, v_h-u_h\right)+a(u_h,v_h-u_h)\geq(f,v_h-u_h),\ v_h\in K^h \tag{20}$$

Now, we apply the $\theta$-scheme on the semi-discrete problem (20); for any $\theta\in[0,1]$ and $k=1,\cdots,n$, we have for $v_h\in K^h$

$$\left(\frac{u_h^k-u_h^{k-1}}{\Delta t}, v_h-u_h^{\theta,k}\right)+a\left(u_h^{\theta,k},v_h-u_h^{\theta,k}\right)\geq\left(f^{\theta,k},v_h-u_h^{\theta,k}\right), \tag{21}$$

where

$$u_h^{\theta,k}=\theta u_h^k+(1-\theta)u_h^{k-1} \tag{22}$$

$$f^{\theta,k}=\theta f(t_k)+(1-\theta)f(t_{k-1}) \tag{23}$$

We have $u_h^{\theta,k}$ that is admissible because

$$u_h^{\theta,k}=\theta u_h^k+(1-\theta)u_h^{k-1}\geq\theta r_h\psi+(1-\theta)r_h\psi=r_h\psi.$$

Thus we can rewrite (21) as: for $u_h^{\theta,k}\in K^h$ and $v_h\in K^h$

$$\left(\frac{u_h^{\theta,k}}{\theta\Delta t}, v_h - u_h^{\theta,k}\right) + a\left(u_h^{\theta,k}, v_h - u_h^{\theta,k}\right) \geq \left(f^{\theta,k} + \frac{u_h^{k-1}}{\theta\Delta t}, v_h - u_h^{\theta,k}\right). \tag{24}$$

Thus, our problem (24) is equivalent to the following coercive discrete elliptic variational inequality:

$$b\left(u_h^{\theta,k}, v_h - u_h^{\theta,k}\right) \geq \left(f^{\theta,k} + \mu u_h^{k-1}, v_h - u_h^{\theta,k}\right),\ v_h \in K^h. \tag{25}$$

Such that

$$b\left(u_h^{\theta,k}, v_h - u_h^{\theta,k}\right) = a\left(u_h^{\theta,k}, v_h - u_h^{\theta,k}\right) + \mu\left(u_h^{\theta,k}, v_h - u_h^{\theta,k}\right). \tag{26}$$

$$\mu = \frac{1}{\theta\Delta t} = \frac{n}{\theta T}.$$

### 3.2. Stability Analysis of $\theta$-Scheme for the P.V.I

The study of the stability of $\theta$-scheme for the American options problem is adapted to [4].

It is possible to analyze stabilitytaking advantage of the structure of eigenvalues of the bilinear form $a(.,.)$, and wecall that $W$ is compactly embedded in $L^2(\Omega)$ since $\Omega$ is bounded.

Let $\{\omega_{ih}\}$ the eigenvectors of $a(.,.)$ form a complete orthonormal basis of $W^h$ in the finite dimensional problem. At each time step $u_h^k \in K^h$, can be expressed $u_h^k$ as well:

$$u_h^k = \sum_{i=1}^{m(h)} u_i^k \omega_{ih} \text{ and } u_i^k = \left(u_h^k, \omega_{ih}\right).$$

Moreover, let $f_h^k$ be the $L^2$-orthogonal projection of $f^{\theta,k}$ into $W^h$, that is, $f_h^k \in W^h$, one has

$$f_h^k = \sum_{i=1}^{m(h)} f_i^k \omega_{ih} \text{ and } f_i^k = \left(f_h^k, \omega_{ih}\right).$$

We are now in a position to prove the stability for $0 \leq \theta < \frac{1}{2}$, choosing in (21) $v = 0$, thus we have for $u_h^{\theta,k} \in K^h$

$$\frac{1}{\Delta t}\left(u_h^k - u_h^{k-1}, u_h^{\theta,k}\right) + a\left(u_h^{\theta,k}, u_h^{\theta,k}\right) \leq \left(f^{\theta,k}, u_h^{\theta,k}\right). \tag{27}$$

For each $i = 1,\cdots,m(h)$, the inequalities (27) is equivalent to

$$\left(\frac{u_i^k - u_i^{k-1}}{\Delta t}\right) + \lambda_{ih}\left(\theta u_i^k + (1-\theta)u_i^{k-1}\right) \leq f_i^k. \tag{28}$$

Since $\omega_{ih}$ are the eigenfunctions means

$$a\left(\omega_{ih}, \omega_{ih}\right) = \lambda_{ih}\left(\omega_{ih}, \omega_{ih}\right) = \lambda_{ih}\delta_{ii} = \lambda_{ih}. \tag{29}$$

If one solves relative to $u_i^k$, we find:

$$u_i^k \leq \frac{1-(1-\theta)\Delta t\lambda_{ih}}{1+\theta\Delta t\lambda_{ih}} u_i^{k-1} + \frac{\Delta t}{1+\theta\Delta t\lambda_{ih}} f_i^k. \tag{30}$$

This inequality system stable if and only if

$$\left|\frac{1-(1-\theta)\Delta t\lambda_{ih}}{1+\theta\Delta t\lambda_{ih}}\right| < 1, \tag{31}$$

that is to say

$$2\theta - 1 > -\frac{2}{\lambda_{ih}\Delta t}, \tag{32}$$

means

$$\Delta t < \frac{2}{(1-2\theta)\lambda_{ih}}. \tag{33}$$

So that this relation satisfied for all the eigenvalues $\lambda_{ih}$ of the bilinear form $a(.,.)$, we have to choose their highest value, we take it for $\lambda_{mh} = \rho(\mathbb{A})$.

**Lemma 1** (Cf. [4]):

For $\theta \ge \frac{1}{2}$ the $\theta$-scheme way is stable unconditionally *i.e.*, stable $\forall \Delta t$.

And if $0 \le \theta < \frac{1}{2}$ the $\theta$-scheme is stable unless

$$\Delta t < \frac{2C}{(1-2\theta)} h^2. \tag{34}$$

With

$$\frac{c_1}{h^2} \le \lambda_{mh} = c_2 h^{-2}. \tag{35}$$

$\lambda_{mh} = \rho(\mathbb{A})$ (spectral radius of $\mathbb{A}$).

Notice that this condition is always satisfied if $0 \le \theta < \frac{1}{2}$. Hence, taking the absolute value of (30), we have

$$\left|u_i^n\right| < \left|u_i^0\right| + \frac{\Delta t}{1+\theta \Delta t \lambda_{ih}} \sum_{k=1}^{n-1} f_i^k, \tag{36}$$

also we deduce that

$$\left\|u_i^n\right\|_\infty < \left\|u_i^0\right\|_\infty + \left\|\frac{\Delta t}{1+\theta \Delta t \lambda_{ih}}\right\|_\infty \sum_{k=1}^{n-1} \left\|f_i^k\right\|_\infty. \tag{37}$$

**Remark 1** (Cf. [4]):

We assume that the coerciveness assumption (19) is satisfied with $\lambda = 0$, and for each $k = 1;\cdots;n$, we find

$$\left\|u_h^k\right\|_2^2 + 2\Delta t \sum_{k=1}^{n} a\left(u_h^{\theta,k}, u_h^{\theta,k}\right) < C(n)\left(\left\|u_h^0\right\|_2^2 + \sum_{k=1}^{n} \Delta t \left\|f^{\theta,k}\right\|_2^2\right). \tag{38}$$

where

$$f^{\theta,k} = \theta f(t_k) + (1-\theta) f(t_{k-1}).$$

## 4. Existence and Uniqueness for Discrete P.V.I

We consider that $u^\infty$ and $u_h^\infty$ are respectively the stationary solutions of the following continue and discrete inequalities:

$$b\left(u^\infty, v - u^\infty\right) \ge \left(f + \lambda u^\infty, v - u^\infty\right). \tag{39}$$

$$b\left(u_h^\infty, v_h - u_h^\infty\right) \ge \left(f + \lambda u_h^\infty, v_h - u_h^\infty\right). \tag{40}$$

where the bilinear form $b(u,v) = a(u,v) + \lambda(u,v)$ satisfies the coercivity condition.

**Theorem 3** (Cf. [9]): Under the previous assumptions, and the maximum principle, there exists a constant $C$ independent of $h$ such that

$$\left\|u^\infty - u_h^\infty\right\|_\infty \le Ch^2 \left|\log h\right|^2.$$

### 4.1. A Fixed Point Mapping Associated with Discrete Problem

We consider the mapping

$$\begin{aligned} T_h &: L_+^\infty(\Omega) \to K^h \\ w &\to T_h(w) = \xi_h, \end{aligned} \tag{41}$$

where $\xi_h$ is the unique solution of the following discrete coercive V.I: find $\xi_h \in K^h$

$$b(\xi_h, v - \xi_h) \ge \left( f^{\theta,k} + \mu w, v - \xi_h \right), \ v \in K^h.$$

**Lemma 2** (Cf. [6]): Under the d.m.p we have if $F \ge \tilde{F}$ then $\xi_h \ge \tilde{\xi}_h$.

**Proposition 1:** Under the previous hypotheses and notations, if we set $\ge \frac{1}{2}$, the mapping $T_h$ is a contraction in $L^\infty(\Omega)$, *i.e.*,

$$\left\| T_h(w) - T_h(\tilde{w}) \right\|_\infty \le \frac{1}{1 + \beta\theta\Delta t} \left\| w - \tilde{w} \right\|_\infty. \tag{42}$$

Therefore, $T_h$ admits a unique fixed point, which coincides with the solution of discrete coercive V.I (25).

**Proof:** For $w \in L^\infty(\Omega)$ and $\tilde{w} \in L^\infty(\Omega)$, we consider $\xi_h = T_h(w) = \partial_h\left( f^{\theta,k} + \mu w, r_h\psi \right)$ (respectively, $\tilde{\xi}_h = T_h(\tilde{w}) = \partial_h\left( f^{\theta,k} + \mu\tilde{w}, r_h\tilde{\psi} \right)$ solution to discrete coercive variational inequality (25) with right-hand side $F^{\theta,k} = f^{\theta,k} + \lambda w$ (respectively $\tilde{F}^{\theta,k} = f^{\theta,k} + \mu\tilde{w}_h$).

Now, set

$$\Phi = \frac{1}{\mu + \beta} \left\| F^{\theta,k} - \tilde{F}^{\theta,k} \right\|_\infty.$$

Since,

$$F^{\theta,k} \le \tilde{F}^{\theta,k} + \left\| F^{\theta,k} - \tilde{F}^{\theta,k} \right\|_\infty.$$

$$F^{\theta,k} \le \tilde{F}^{\theta,k} + \frac{a_0 + \mu}{\mu + \beta} \left\| F^{\theta,k} - \tilde{F}^{\theta,k} \right\|_\infty.$$

(because $a_0 \ge \beta > 0$).

$$F^{\theta,k} \le \tilde{F}^{\theta,k} + (a_0 + \mu)\Phi.$$

So using Lemma 2 gives

$$\partial_h\left( F^{\theta,k}, r_h\psi \right) \le \partial_h\left( \tilde{F}^{\theta,k} + (a_0 + \mu)\Phi, r_h\tilde{\psi} \right).$$

On the other hand, one has

$$\partial_h\left( \tilde{F}^{\theta,k}, r_h\tilde{\psi} \right) + \Phi = \partial_h\left( \tilde{F}^{\theta,k} + (a_0 + \mu)\Phi, r_h\tilde{\psi} + \Phi \right).$$

Indeed, $\tilde{\xi}_h + \Phi$ is solution of

$$b\left( \tilde{\xi}_h + \Phi, v + \Phi - \left( \tilde{\xi}_h + \Phi \right) \right) \ge \left( \tilde{F}^{\theta,k} + (a_0 + \mu)\Phi, v + \Phi - \left( \tilde{\xi}_h + \Phi \right) \right)$$

$$\tilde{\xi}_h + \Phi \le r_h\tilde{\psi} + \Phi, \ \ v + \Phi \le r_h\tilde{\psi} + \Phi, \ \ \forall v \in K^h,$$

thus

$$\partial_h\left( F^{\theta,k}, r_h\psi \right) \le \partial_h\left( \tilde{F}^{\theta,k} + (a_0 + \mu)\Phi, r_h\tilde{\psi} \right) \le \partial_h\left( \tilde{F}^{\theta,k} + (a_0 + \mu)\Phi, r_h\tilde{\psi} + \Phi \right).$$

Therefore

$$\xi_h \le \tilde{\xi}_h + \Phi.$$

Similarly, interchanging the roles of $w$ and $\tilde{w}$ we also get

$$\tilde{\xi}_h \le \xi_h + \Phi.$$

Consequently,

$$\begin{aligned}\left\|T_h(w)-T_h(\tilde{w})\right\|_\infty &\le \frac{1}{\mu+\beta}\left\|F^{\theta,k}-\tilde{F}^{\theta,k}\right\|_\infty \\ &\le \frac{1}{\mu+\beta}\left\|f^{\theta,k}+\mu w-\left(f^{\theta,k}+\mu\tilde{w}\right)\right\|_\infty \\ &\le \frac{\mu}{\mu+\beta}\left\|w-\tilde{w}\right\|_\infty .\end{aligned}$$

$$\left\|T_h(w)-T_h(\tilde{w})\right\|_\infty \le \frac{1}{1+\beta\theta\Delta t}\left\|w-\tilde{w}\right\|_\infty$$

which is the desired result.

**Remark 2:** If we set $0\le\theta<\frac{1}{2}$, the mapping $T_h$ is a contraction in $L^\infty(\Omega)$, *i.e.*,

$$\left\|T_h(w)-T_h(\tilde{w})\right\|_\infty \le \frac{2Ch^2}{2Ch^2+\beta\theta(1-2\theta)}\left\|w-\tilde{w}\right\|_\infty . \tag{43}$$

Therefore, $T_h$ admits a unique fixed point, which coincides with the solution of discrete coercive V.I (25).

**Proof:** Under condition of stability, we have shown the $\theta$-scheme is stable if and only if $\Delta t<\frac{2C}{1-2\theta}h^2$, thus it can be easily show that

$$\begin{aligned}\left\|T_h(w)-T_h(\tilde{w})\right\|_\infty &\le \frac{1}{1+\beta\theta\Delta t}\left\|w-\tilde{w}\right\|_\infty \le \frac{1}{1+\beta\theta\left(\frac{1-2\theta}{2Ch^2}\right)}\left\|w-\tilde{w}\right\|_\infty \\ &= \frac{2Ch^2}{2Ch^2+\beta\theta(1-2\theta)}\left\|w-\tilde{w}\right\|_\infty ,\end{aligned}$$

also it can be found that

$$\left\|T_h(w)-T_h(\tilde{w})\right\|_\infty \le \frac{2Ch^2}{2Ch^2+\beta\theta(1-2\theta)}\left\|w-\tilde{w}\right\|_\infty$$

which is the desired result.

## 4.2. Iterative Discrete Algorithm

We choose $u_h^0$ as the solution of the following discrete equation

$$b\left(u_h^0,v\right)=\left(g^0,v\right),\quad v\in V^h, \tag{44}$$

where $g^0$ is a regular function given.

Now we give our following discrete algorithm

$$u_h^{\theta,k}=T_h u_h^{k-1},\quad k=1,\cdots,n,\quad u_h^{\theta,k}\in V^h, \tag{45}$$

where $u_h^{\theta,k}$ is the solution of the problem (25).

**Remark 3** cf. [7]: If we choose $\theta=1$ in (45) we get Bensoussan's algorithm.

**Proposition 2:** Under the previous hypotheses and notations, we have the following estimate of convergence $\theta\ge\frac{1}{2}$

$$\left\|u_h^{\theta,k}-u_h^\infty\right\|_\infty \le \left(\frac{1}{1+\beta\theta\Delta t}\right)^k\left\|u_h^\infty-u_h^0\right\|_\infty . \tag{46}$$

And if $0\le\theta<\frac{1}{2}$

$$\left\|u_h^{\theta,k}-u_h^{\infty}\right\|_{\infty}\le\left(\frac{2Ch^2}{2Ch^2+\beta\theta(1-2\theta)}\right)^k\left\|u_h^{\infty}-u_h^0\right\|_{\infty}. \tag{47}$$

**Proof:** We set a first case $\theta\ge\frac{1}{2}$, and we have

$$u_h^{\infty}=T_h u_h^{\infty},$$

$$\left\|u_h^{\theta,1}-u_h^{\infty}\right\|_{\infty}=\left\|T_h u_h^0-T_h u_h^{\infty}\right\|_{\infty}\le\frac{1}{1+\beta\theta\Delta t}\left\|u_h^0-u_h^{\infty}\right\|_{\infty}$$

We assume that

$$\left\|u_h^{\theta,k}-u_h^{\infty}\right\|_{\infty}\le\left(\frac{1}{1+\beta\theta\Delta t}\right)^k\left\|u_h^0-u_h^{\infty}\right\|_{\infty},$$

so

$$\left\|u_h^{\theta,k+1}-u_h^{\infty}\right\|_{\infty}=\left\|T_h u_h^k-T_h u_h^{\infty}\right\|_{\infty}\le\frac{1}{1+\beta\theta\Delta t}\left\|u_h^k-u_h^{\infty}\right\|_{\infty}$$

thus

$$\left\|u_h^{\theta,k+1}-u_h^{\infty}\right\|_{\infty}\le\left(\frac{1}{1+\beta\theta\Delta t}\right)^{k+1}\left\|u_h^0-u_h^{\infty}\right\|_{\infty}$$

For a second case $0\le\theta<\frac{1}{2}$ one can easily show that

$$\left\|u_h^{\theta,k}-u_h^{\infty}\right\|_{\infty}\le\left(\frac{2Ch^2}{2Ch^2+\beta\theta(1-2\theta)}\right)^k\left\|u_h^0-u_h^{\infty}\right\|_{\infty}$$

which is the desired result.

## 5. Asymptotic Behavior

This section is devoted to the proof of principal result of the present paper, where we prove the theorem of the asymptotic behavior in $L^{\infty}$-norm for parabolic variational inequalities.

Now, we evaluate the variation in $L^{\infty}$ between $u_h^{\infty}(x,T)$, the discrete solution calculated at the moment $T=n\Delta t$ and $u^{\infty}$, the continuous solution of (39).

**Theorem 4:** (The principal result). Under conditions of Theorem (3) and Proposition (2), we have for the first case $\theta\ge\frac{1}{2}$

$$\left\|u_h^{\theta,n}-u^{\infty}\right\|_{\infty}\le C\left[h^2\left|\log h\right|^2+\left(\frac{1}{1+\beta\theta\Delta t}\right)^n\right], \tag{48}$$

and for the second case $0\le\theta<\frac{1}{2}$

$$\left\|u_h^{\theta,n}-u^{\infty}\right\|_{\infty}\le C\left[h^2\left|\log h\right|^2+\left(\frac{2Ch^2}{2Ch^2+\beta\theta(1-2\theta)}\right)^n\right], \tag{49}$$

where $C$ is a constant independent of $h$ and $k$.

**Proof:** We have

$$u_h^{\theta,k}=u_h(x,t)\ \text{ for } t\in\left]\left(k-1\right)\Delta t;k\Delta t\right[,$$

thus

$$u_h^{\theta,n}(x) = u_h(x,T)$$

Then

$$\left\|u_h(T) - u^\infty\right\|_\infty = \left\|u_h^{\theta,n} - u^\infty\right\|_\infty \le \left\|u_h^{\theta,n} - u_h^\infty\right\|_\infty + \left\|u_h^\infty - u^\infty\right\|_\infty.$$

Using the Theorem (3) and the Proposition (2), we have for $\theta \ge \frac{1}{2}$

$$\left\|u_h^{\theta,n} - u^\infty\right\|_\infty \le C\left[h^2\left|\log h\right|^2 + \left(\frac{1}{1+\beta\theta\Delta t}\right)^n\right],$$

and for $0 \le \theta < \frac{1}{2}$ we have

$$\left\|u_h^{\theta,n} - u^\infty\right\|_\infty \le C\left[h^2\left|\log h\right|^2 + \left(\frac{2Ch^2}{2Ch^2 + \beta\theta(1-2\theta)}\right)^n\right].$$

## 6. Perspective

In the following, we will consolidate our theoretical results by numerical simulation, which allows us to locate the free boundary, a very interesting thing in practice to calculate the price of the American options.

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