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Exact Tail Asymptotics for a Queueing System with a Retrial Orbit and Batch Service

Huijun Lu

Huling Middle School, Yan'an, China
Email: 1187588535@qq.com

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Abstract

This paper discusses a queueing system with a retrial orbit and batch service, in which the quantity of customers' rooms in the queue is finite and the space of retrial orbit is infinite. When the server starts serving, it serves all customers in the queue in a single batch, which is the so-called batch service. If a new customer or a retrial customer finds all the customers' rooms are occupied, he will decide whether or not to join the retrial orbit. By using the censoring technique and the matrix analysis method, we first obtain the decay function of the stationary distribution for the quantity of customers in the retrial orbit and the quantity of customers in the queue. Then based on the form of decay rate function and the Karamata Tauberian theorem, we finally get the exact tail asymptotics of the stationary distribution.

Keywords

Exact Tail Asymptotics, Batch Service, Censoring Technique, Matrix Analysis Method, Karamata Tauberian Theorem

1. Introduction

Due to the complexity of the queueing system with a retrial orbit and batch service, we cannot obtain the exact joint stationary distribution for the quantity of customers in the orbit and the quantity of customers in the queue. Therefore, we study the tail asymptotic behavior of the joint stationary distribution, and we get an intuitive conclusion.

The so-called retrial orbit is the retrial queue which has been taken to model many problems in real life, such as computer systems, telephone systems and communication networks. Retrial queues mean that a customer who finds that there is no space in the system for him to receive service will join the retrial orbit.

Then the retrial customer (customer in the retrial orbit) repeatedly tries to get into the queue or server space to obtain the service. There are a large amount of documents on the retrial queues, including articles [1] [2] [3], books [4] [5] and so on. Some scholars have studied the topic of the tail asymptotic behavior for the stationary distribution of the queue size in the retrial queues. The research on this topic can be found in the following articles. Shang *et al.* [6] studied the tail asymptotic behavior of the stable queue in the $M/G/1$ retrial queueing system. By using the analytic properties of probability generating functions, Kim *et al.* [7], Kim *et al.* [8], and Kim *et al.* [9] considered different retrial queues and got the tail asymptotic property of the queue size (the number of retrial customers) distribution. By using the censoring technique and matrix analysis method, Liu and Zhao [10] and Liu *et al.* [11] studied the $M/M/c$ retrial queueing system with different conditions and obtained the asymptotic lower and upper bounds of the stationary distribution. Then Kim *et al.* [12] and Kim *et al.* [13] improved the results of [10] [11] and got the more accurate tail asymptotic results. Furthermore, Liu and Zhao [14] used the random decomposition method to study the retrial queue with two types of customers and obtained the asymptotic of the tail probability.

In this paper, we focus on the batch service which depends on the length of the customer queue. Batch service queueing system has been studied by many scholars. Oyen *et al.* [15], Wal *et al.* [16] and Boxma *et al.* [17] studied the batch-service polling system. In recent years, Bountali and Economou [18] and Ommeren *et al.* [19] considered batch service queueing systems and assumed that the service time is independent of the batch size. However, Pradhan and Gupta [20] and Du *et al.* [21] studied the batch service which depends on the batch size.

The paper is organized as follows. In Section 2, we first describe the queueing system with a retrial orbit and batch service, and then obtain some useful equations by using the censoring technique and the matrix analysis method. In Section 3, based on the asymptotic form of the rate matrix, the expression of the decay function about the stationary distribution is obtained. In Section 4, we rewrite the infinitesimal generator Q and get the exact tail asymptotics for the stationary distribution. The conclusion is made in Section 5.

2. Model Description and Analysis

We first introduce the model of this single server queueing system with batch service and a retrial orbit. The maximum quantity of customers' rooms in the system queue is N. When the server starts serving, all customers in the queue will be served as a batch and will leave the system after completing the service. The service time intervals are exponential random variables with the parameter

$$\mu_L = \begin{cases} 0, & L = 0, \\ (N-L)\mu, & L = 1, 2, \cdots, N, \end{cases}$$

where μ is a positive constant, and L implies the quantity of customers in the queue. The customers' arrival process is a Poisson process with the rate λ. The first customer waiting in the queue may leave the system due to he is impatient after a random waiting time which complies with an exponential distribution with the rate θ. If a new customer arrives at the system and finds one empty room, the customer will join the queue's room. Otherwise, all rooms are occupied, the customer will join the retrial orbit with probability p and become a retrial customer, or will leave the system immediately with probability $1-p$. Each retrial customer in the orbit frequently tries for service according to a Poisson process with the rate α until he finds an empty room based upon retrial, and then joins the empty room. If a retrial customer finds all rooms are occupied, the customer will return to the retrial orbit with probability q, or will leave the system with probability $\bar{q}=1-q$. In this paper, we assume that $\bar{q}>0$, and find that the system is stable without any conditions.

Let $N(t)$ be the quantity of customers in the retrial orbit at time t, and let $I(t)$ be the quantity of customers in the queue at time t. $\{(N(t),I(t)):t\ge 0\}$ is a continuous-time Markov chain with the state space

$$\Omega=\{(n,i):n=0,1,\cdots;i=0,1,\cdots,N\}.$$

Based on the model described above, Q matrix of the Markov chain has the structure of QBD (quasi-birth-and-death) as follows

$$Q=\begin{bmatrix} B_0 & A & & & & & \\ C_1 & B_1 & A & & & & \\ & C_2 & B_2 & A & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & C_n & B_n & A & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix},$$

where

$$A=\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & \lambda p \end{bmatrix},\ C_n=\begin{bmatrix} 0 & n\alpha & & & \\ & 0 & n\alpha & & \\ & & \ddots & \ddots & \\ & & & 0 & n\alpha \\ & & & & n\alpha\bar{q} \end{bmatrix},$$

$$B_n=\begin{bmatrix} -(\lambda+n\alpha) & \lambda & & & & \\ (N-1)\mu+\theta & \beta_1 & \lambda & & & \\ (N-2)\mu & \theta & \beta_2 & \lambda & & \\ \vdots & & \ddots & \ddots & \ddots & \\ \mu & & & \theta & \beta_{N-1} & \lambda \\ 0 & & & & \theta & -(\lambda p+\theta+n\alpha\bar{q}) \end{bmatrix},$$

and

$$\beta_i = -\left(\lambda + \left(N - i\right)\mu + \theta + n\alpha\right), \quad i = 1,2,\cdots,N-1; \quad n = 0,1,\cdots.$$

We assume that $\boldsymbol{\pi} = \left(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \cdots\right)$ is the stationary probability vector of the Markov chain, where $\boldsymbol{\pi}_n = \left(\pi_{n,0}, \pi_{n,1}, \cdots, \pi_{n,N}\right), n \geq 0$, and

$$\pi_{n,i} = \lim_{t\to\infty} P\left(N\left(t\right) = n, I\left(t\right) = i\right), \quad i = 0,1,\cdots,N.$$

We can find that $\pi_{n,i}$ represents the joint stationary probability of the quantity of customers in the retrial orbit and the queue length. Next, we define that $Q_0 = Q$. Moreover, Q_n $(n = 1,2,\cdots)$ is defined as the submatrix which is obtained by removing the element matrixes of the first n rows and the first n columns in the Q, where (i, j)th element of Q_n is as follows

$$\left(Q_n\right)_{i,j} = \begin{cases} A, & \text{if } j = i+1, i = 1,2,\cdots, \\ B_{n+i}, & \text{if } j = i, i = 0,1,2,\cdots, \\ C_{n+i}, & \text{if } j = i-1, i = 1,2,\cdots. \end{cases}$$

According to the matrix analysis method, $\boldsymbol{\pi}_n$ has the solution of the following matrix form

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 R_1 R_2 \cdots R_n, \quad n = 1,2,\cdots, \tag{1}$$

where $\boldsymbol{\pi}_0$ is the solution to the matrix equation below

$$\boldsymbol{\pi}_0 \left(B_0 + R_1 C_1\right) = 0, \tag{2}$$

and

$$R_n = A\hat{Q}_n\left(1,1\right), \quad n = 1,2,\cdots, \tag{3}$$

where $\hat{Q}_n = \left(-Q_n\right)^{-1}$ and $\hat{Q}_n\left(1,1\right)$ represents the submatrix of the first row and the first column in $\hat{Q}_n$.

Throughout the paper, we let $\boldsymbol{e}_{N+1} = \left(0,0,\cdots,0,1\right)^{\mathrm{T}}$. Based on the special structure of A (only the element in the lower right-hand corner is non-zero) and combined with (3), we can obtain that R_n also has the special structure as follows

$$R_n = \boldsymbol{e}_{N+1}\boldsymbol{r}_n,$$

where $\boldsymbol{r}_n = \left(r_{n,0}, r_{n,1}, \cdots, r_{n,N}\right)$. Combining formula (1) with the structure of R_n, we can get

$$\boldsymbol{\pi}_n = \varphi_n \boldsymbol{\pi}_0 \boldsymbol{e}_{N+1} \boldsymbol{r}_n,$$

where $\varphi_n = \prod_{j=1}^{n-1} r_{j,N}$. Then, we have

$$\boldsymbol{\pi}_n = \pi_{0,N}\varphi_n \boldsymbol{r}_n, \quad n = 1,2,\cdots. \tag{4}$$

$\boldsymbol{\pi}_0$ is uniquely determined by (2) and the normalization conditions

$$\sum_{n=0}^{\infty} \boldsymbol{\pi}_n \boldsymbol{e} = \boldsymbol{\pi}_0 \sum_{n=0}^{\infty} \left(\prod_{i=1}^{n} R_i\right) \boldsymbol{e} = 1.$$

For studying this queuing system, we need to introduce the censored matrix $Q^{\leq(n-1)}$ with censoring set $S_{\leq(n-1)} = \left\{\left(s,i\right); s = 0,1,\cdots,n-1; i = 0,1,\cdots,N\right\}$. Based on the censoring technique, we can obtain the (i, j)th element of $Q^{\leq(n-1)}$

$$\left(Q^{\leq(n-1)}\right)_{i,j} = \begin{cases} A, & \text{if } j=i+1, i=0,1,\cdots,n-2, \\ B_i, & \text{if } j=i, i=0,1,\cdots,n-2, \\ C_i, & \text{if } j=i-1, i=1,2,\cdots,n-1, \\ B_{n-1}+R_nC_n, & \text{if } j=i=n-1, \end{cases}$$

where

$$R_nC_n = n\alpha \boldsymbol{e}_{N+1}\left(0, r_{n,0}, r_{n,1}, \cdots, r_{n,N-2}, r_{n,N-1}+\bar{q}r_{n,N}\right). \tag{5}$$

According to the sum of all the rows of censored matrix $Q^{\leq(n-1)}$ being zero, we know that

$$C_{n-1}+B_{n-1}+R_nC_n=0.$$

After expanding the last row of the above equation, we can obtain the following key equation

$$\sum_{i=0}^{N-1} r_{n,i}+\bar{q}r_{n,N}=\frac{\lambda p}{n\alpha}, \quad n=1,2,\cdots. \tag{6}$$

Next, we discuss the censored matrix $Q^{(n)}$. From the definition of the censored matrix, we know that $Q^{(n)}=\left(Q^{\leq n}\right)^{(n)}$. By using the censoring technique (see Liu and Zhao [10]), and based on the special structure of A and C_n, we have

$$Q^{(n)}=\left(B_n+R_{n+1}C_{n+1}\right)+\begin{bmatrix} 0 & 0 & \cdots & x_0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{N-1} \\ 0 & 0 & \cdots & x_N \end{bmatrix}.$$

As $Q^{(n)}$ is also the infinitesimal generator, the respective sum of each row is zero. Combined with (5), the exact expressions of x_i ($i=0,1,2,\cdots,N$) can be obtained. We find that

$$\begin{cases} x_i=n\alpha, & i=0,1,\cdots,N-1 \\ x_N=n\alpha\bar{q}. \end{cases}$$

After the above analysis of $Q^{(n)}$, we have

$$Q^{(n)}=\begin{bmatrix} -(\lambda+n\alpha) & \lambda & & & & n\alpha \\ (N-1)\mu+\theta & \beta_1 & \lambda & & & n\alpha \\ (N-2)\mu & \theta & \beta_2 & \lambda & & n\alpha \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \mu & & & \theta & \beta_{N-1} & \lambda+n\alpha \\ 0 & a_{n+1,0} & a_{n+1,1} & \cdots & \theta+a_{n+1,N-2} & \chi_{n+1} \end{bmatrix},$$

where

$$\chi_{n+1}=-(\lambda p+\theta)+(n+1)\alpha\left(r_{n+1,N-1}+\bar{q}r_{n+1,N}\right),$$

$$a_{n+1,i}=(n+1)\alpha r_{n+1,i}, \quad i=0,1,\cdots,N-1.$$

Based on the balance equation of the censored matrix and combined with (4), we can get

$$\boldsymbol{r}_nQ^{(n)}=0, \quad n=0,1,\cdots.$$

Continue to expand the above matrix equation, we have the following equations

$$\begin{cases} -(\lambda+n\alpha)r_{n,0}+\theta r_{n,1}+\sum_{i=1}^{N-1}(N-i)\mu r_{n,i}=0 & (7)\\ \lambda r_{n,0}+\beta_1 r_{n,1}+\theta r_{n,2}+a_{n+1,0}r_{n,N}=0 & (8)\\ \lambda r_{n,1}+\beta_2 r_{n,2}+\theta r_{n,3}+a_{n+1,1}r_{n,N}=0 & (9)\\ \vdots \\ \lambda r_{n,N-2}+\beta_{N-1}r_{n,N-1}+\left[\theta+a_{n+1,N-2}\right]r_{n,N}=0 & (10)\\ n\alpha\sum_{i=0}^{N-1}r_{n,i}+\lambda r_{n,N-1}+\chi_{n+1}r_{n,N}=0 \end{cases}$$

3. Decay Function of $\pi_{n,i}$

In this queueing system, $\pi_{n,i}$ ($n>0$) does not have the exact expression when N takes a general value. Therefore, we need to focus on the asymptotic property of $\pi_{n,i}$ when $n\to 0$. We discuss the decay function of $\pi_{n,i}$ to study the asymptotic behavior.

We assume that a decay function of $\pi_{n,i}$ is $h_i(n)>0$. For each $i>0$, we have

$$0<M_i\le\liminf_{n\to\infty}\frac{\pi_{n,i}}{h_i(n)}\le\limsup_{n\to\infty}\frac{\pi_{n,i}}{h_i(n)}\le N_i,$$

where M_i and N_i are two constants independent of *n*. That is, for each *i*, there are always two positive constants M_i and N_i existing independent of *n*, satisfying when $n\to 0$

$$M_i h_i(n)\le\pi_{n,i}\le N_i h_i(n).$$

In order to find the decay function $h_i(n)$, we need to analyze the asymptoticity of $r_{n,i}$ in the following theorem and corollary. We define $o(x_n)$ and $O(x_n)$ as $\lim_{n\to\infty}o(x_n)/x_n=0$ and $\lim_{n\to\infty}O(x_n)/x_n=W\ne 0$ respectively, where W is a constant.

Theorem 3.1. *For* $i=0,1,2,\cdots,N$, *we have*

$$r_{n,i}=\frac{\lambda p}{\alpha\bar{q}}\cdot\frac{1}{n^{N-i+1}}\left(\frac{\theta}{\alpha}\right)^{N-i}+o\left(\frac{1}{n^{N-i+1}}\right).$$

Proof. Based on (6), when $n\to\infty$, we get that

$$r_{n,i}\to 0,\quad i=0,1,\cdots,N. \tag{11}$$

According to (7) and (11), we can get $nr_{n,0}\to 0$. According to (8) and the above conclusions, we have $nr_{n,1}\to 0$. Similarly, according to (9) and the above conclusions, we obtain that $nr_{n,2}\to 0$. Finally, we get $nr_{n,N-1}\to 0$. That is,

$$nr_{n,i}\to 0,\quad i=0,1,\cdots,N-1. \tag{12}$$

We can know that $n\left(\sum_{i=0}^{N-1}r_{n,i}+\bar{q}r_{n,N}\right)=\frac{\lambda p}{\alpha}$ from the result of (6). Next, substituting the conclusion of (12) into it, we can obtain

$$nr_{n,N} \to \frac{\lambda p}{\alpha \bar{q}},$$

or

$$r_{n,N} = \frac{\lambda p}{\alpha \bar{q}} \frac{1}{n} \left(\frac{\theta}{\alpha}\right)^0 + o\left(\frac{1}{n}\right).$$

Obviously, multiply both sides of (7)-(10) by *n*, let $n \to \infty$, we have

$$n^2 r_{n,i} \to 0, \quad i = 0,1,\cdots,N-2,$$

and

$$n^2 r_{n,N-1} \to \frac{\lambda p}{\alpha \bar{q}} \cdot \frac{\theta}{\alpha},$$

that is

$$r_{n,N-1} = \frac{\lambda p}{\alpha \bar{q}} \cdot \frac{1}{n^2} \left(\frac{\theta}{\alpha}\right) + o\left(\frac{1}{n^2}\right).$$

Repeat the above process, multiply the equations of (7)-(10) by n^2, we can get $n^3 r_{n,i} \to 0$, $i = 0,1,\cdots,N-3$, and

$$n^3 r_{n,N-2} \to \frac{\lambda p}{\alpha \bar{q}} \cdot \left(\frac{\theta}{\alpha}\right)^2,$$

or

$$r_{n,N-2} = \frac{\lambda p}{\alpha \bar{q}} \cdot \frac{1}{n^3} \left(\frac{\theta}{\alpha}\right)^2 + o\left(\frac{1}{n^3}\right).$$

By analogy with the same way, finally we get

$$r_{n,i} = \frac{\lambda p}{\alpha \bar{q}} \cdot \frac{1}{n^{N-i+1}} \left(\frac{\theta}{\alpha}\right)^{N-i} + o\left(\frac{1}{n^{N-i+1}}\right), \quad i = 0,1,\cdots,N.$$

The proof is finished. □

The asymptotic result in Theorem 3.1 can replace "*o*" with "*O*" to improve the asymptotic formula.

Corollary 3.1. *For* $i = 0,1,2,\cdots,N$, *we have*

$$r_{n,i} = \frac{\lambda p}{\alpha \bar{q}} \frac{1}{n^{N-i+1}} \left(\frac{\theta}{\alpha}\right)^{N-i} + O\left(\frac{1}{n^{N-i+2}}\right). \quad (13)$$

Proof. According to (6) and Theorem 3.1, we can get

$$r_{n,N} = \frac{\lambda p}{\alpha \bar{q}} \frac{1}{n} \left(\frac{\theta}{\alpha}\right)^0 + O\left(\frac{1}{n^2}\right).$$

Based on (10), we have

$$r_{n,N-1} = \frac{\lambda p}{\alpha \bar{q}} \frac{1}{n^2} \left(\frac{\theta}{\alpha}\right) + O\left(\frac{1}{n^3}\right).$$

Similarly, according to the above methods, when $i = 0,1,\cdots,N$, the result can be derived as follows

$$r_{n,i}=\frac{\lambda p}{\alpha\bar{q}}\frac{1}{n^{N-i+1}}\left(\frac{\theta}{\alpha}\right)^{N-i}+O\left(\frac{1}{n^{N-i+2}}\right).$$

The conclusion is proved. □

Next, we further improve the asymptotic expression. For example, we can find that

$$r_{n,N}=\frac{\lambda p}{n\alpha\bar{q}}-\frac{1}{\bar{q}}\sum_{i=0}^{N-1}r_{n,i}. \tag{14}$$

After substituting (13) into it, we obtain that

$$r_{n,N}=\frac{\lambda p}{\alpha\bar{q}}\frac{1}{n}\left(\frac{\theta}{\alpha}\right)^{0}-\frac{\lambda p}{\alpha\bar{q}}\frac{1}{n^2}\left(\frac{\theta}{\alpha}\right)\cdot\frac{1}{\bar{q}}+O\left(\frac{1}{n^3}\right). \tag{15}$$

Based on the asymptotic expression of $r_{n,0}$ in (15), combining (10), we can get

$$r_{n,N-1}=\frac{\lambda p}{\alpha\bar{q}}\frac{1}{n^2}\left(\frac{\theta}{\alpha}\right)-\frac{\lambda p}{\alpha\bar{q}}\frac{1}{n^3}\left(\frac{\theta}{\alpha}\right)^2\cdot\left(\frac{1}{\bar{q}}+\frac{\lambda+\mu+\theta}{\theta}\right)+O\left(\frac{1}{n^4}\right).$$

From (13), it is easy to get the next formula

$$r_{n,N-2}=\frac{\lambda p}{\alpha\bar{q}}\frac{1}{n^3}\left(\frac{\theta}{\alpha}\right)^2+O\left(\frac{1}{n^4}\right).$$

Substituting the above results of $r_{n,N-1}$ and $r_{n,N-2}$ into (14), after a simple calculation, we can obtain

$$r_{n,N}=\frac{\lambda p}{\alpha\bar{q}}\frac{1}{n}\left[1-\frac{1}{n}\left(\frac{\theta}{\alpha}\right)\cdot\frac{1}{\bar{q}}+\frac{1}{n^2}\left(\frac{\theta}{\alpha}\right)^2\cdot\frac{1}{\bar{q}}\cdot\left(\frac{1}{\bar{q}}+\frac{\lambda+\mu}{\theta}\right)+O\left(\frac{1}{n^3}\right)\right].$$

The above formula can be conveyed as follows

$$r_{n,N}=\frac{\lambda p}{n\alpha\bar{q}}\left[1-\frac{a}{n}+\frac{b}{n^2}+O\left(\frac{1}{n^3}\right)\right], \tag{16}$$

where $a=\dfrac{\theta}{\alpha\bar{q}}$, $b=a^2\left(1+\dfrac{\lambda\bar{q}+\mu\bar{q}}{\theta}\right)$. From the above asymptotic results of $r_{n,i}$, we obtain the decay function of $\pi_{n,i}$ in the below theorem.

Theorem 3.2. *In the queueing system with a retrial orbit and batch service, the decay function* $h_i(n)$ *of the stationary probability* $\pi_{n,i}$ *is*

$$h_i(n)=\frac{1}{n!}\left(\frac{\lambda p}{\alpha\bar{q}}\right)^n\cdot n^{-\frac{\theta}{\alpha\bar{q}}-N+i},\quad i=0,1,2,\cdots,N. \tag{17}$$

Proof. As a and b in (16) have the same expression, the condition $a^2-4b<0$ is satisfied, and there always exists two positive real numbers b_1 and b_2, satisfying $b_1\le b\le b_2$ so that $a^2-4b_u<0, u=1,2$.

According to the expression of $r_{n,N}$ in (16), there always exists a positive integer N_0. For all $n>N_0$, we have

$$0<\frac{\lambda p}{n\alpha\bar{q}}\left(1-\frac{a}{n}+\frac{b_1}{n^2}\right)<r_{n,N}<\frac{\lambda p}{n\alpha\bar{q}}\left(1-\frac{a}{n}+\frac{b_2}{n^2}\right).$$

Therefore, when $n>N_0$, we get

$$0<U^0\left(\frac{\lambda p}{\alpha\bar{q}}\right)^n\prod_{j=N_0+1}^{n}\frac{1}{j}\left(1-\frac{a}{j}+\frac{b_1}{j^2}\right)<\prod_{j=1}^{n}r_{j,N}<U^0\left(\frac{\lambda p}{\alpha\bar{q}}\right)^n\prod_{j=N_0+1}^{n}\frac{1}{j}\left(1-\frac{a}{j}+\frac{b_2}{j^2}\right),$$

where $U^0=\left(\frac{\lambda p}{\alpha\bar{q}}\right)^{-N_0}\prod_{j=1}^{N_0}r_{j,N}$ is a constant independent of *n*. In accordance with Corollary 3.2 of Liu *et al.* [11], we can get the asymptotic results of the upper and lower bounds of $r_{1,N}r_{2,N}\cdots r_{n,N}$, as follows

$$U^0\left(\frac{\lambda p}{\alpha\bar{q}}\right)^n\prod_{j=N_0+1}^{n}\frac{1}{j}\left(1-\frac{a}{j}+\frac{b_1}{j^2}\right)\sim U^{(1)}\frac{1}{n!}\left(\frac{\lambda p}{\alpha\bar{q}}\right)^n\cdot n^{-\frac{\theta}{\alpha\bar{q}}},$$

$$U^0\left(\frac{\lambda p}{\alpha\bar{q}}\right)^n\prod_{j=N_0+1}^{n}\frac{1}{j}\left(1-\frac{a}{j}+\frac{b_2}{j^2}\right)\sim U^{(2)}\frac{1}{n!}\left(\frac{\lambda p}{\alpha\bar{q}}\right)^n\cdot n^{-\frac{\theta}{\alpha\bar{q}}},$$

where, $U^{(1)}$ and $U^{(2)}$ are positive constants independent of *n*.

Then due to $\pi_{n,N}=\pi_{0,N}\prod_{j=1}^{n}r_{j,N}$, we have

$$\pi_{0,N}U^{(1)}\frac{1}{n!}\left(\frac{\lambda p}{\alpha\bar{q}}\right)^n\cdot n^{-\frac{\theta}{\alpha\bar{q}}}<\pi_{n,N}<\pi_{0,N}U^{(2)}\frac{1}{n!}\left(\frac{\lambda p}{\alpha\bar{q}}\right)^n\cdot n^{-\frac{\theta}{\alpha\bar{q}}},$$

Let $U_0=\pi_{0,N}U^{(1)}$, $U_0'=\pi_{0,N}U^{(2)}$, then

$$h_N(n)=\frac{1}{n!}\left(\frac{\lambda p}{\alpha\bar{q}}\right)^n\cdot n^{-\frac{\theta}{\alpha\bar{q}}}.$$

From (4), we know $\pi_{n,i}=\pi_{n,N}\frac{r_{n,i}}{r_{n,N}}$, $i=0,1,\cdots,N-1$. Then substituting (13) and (16) into it, we can get

$$h_i(n)=\frac{1}{n!}\left(\frac{\lambda p}{\alpha\bar{q}}\right)^n\cdot n^{-\frac{\theta}{\alpha\bar{q}}-N+i},\quad i=0,1,\cdots,N-1.$$

The conclusion is proved. □

4. Exact Tail Asymptotics for $\pi_{n,i}$

The infinitesimal generator Q of the Markov chain is partitioned in conformity with the level, whose (*i*, *j*)th element can be written as

$$\left(\tilde{Q}\right)_{i,j}=\begin{cases}\tilde{A}, & \text{if } j=i+1, i=0,1,2,\cdots,\\ \tilde{B}_0+i\tilde{B}_i, & \text{if } j=i, i=0,1,2,\cdots,\\ i\tilde{C}, & \text{if } j=i-1, i=1,2,\cdots,\end{cases}$$

where

$$\tilde{A}=\begin{bmatrix}0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & \lambda p\end{bmatrix},\tilde{C}=\begin{bmatrix}0 & \alpha & & & \\ & 0 & \alpha & & \\ & & \ddots & \ddots & \\ & & & 0 & \alpha \\ & & & & \alpha\bar{q}\end{bmatrix},$$

$$\tilde{B}_1 = \begin{bmatrix} -\alpha & & & & & \\ & -\alpha & & & & \\ & & -\alpha & & & \\ & & & \ddots & & \\ & & & & -\alpha & \\ & & & & & -\alpha\bar{q} \end{bmatrix}, \tilde{B}_0 = \begin{bmatrix} -\lambda & \lambda & & & & \\ \eta_1+\theta & \kappa_1 & \lambda & & & \\ \eta_2 & \theta & \kappa_2 & \lambda & & \\ \vdots & & \ddots & \ddots & \ddots & \\ \eta_{N-1} & & & \theta & \kappa_{N-1} & \lambda \\ 0 & & & & \theta & -(\lambda p+\theta) \end{bmatrix},$$

and

$$\kappa_i = -\left(\lambda + (N-i)\mu + \theta\right),$$
$$\eta_i = (N-i)\mu, \quad i = 1,2,\cdots,N-1.$$

Throughout the paper, we define $x_n \sim y_n$ when $n \to \infty$, which denotes $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$ where $\{x_n : n = 0,1,\cdots\}$ and $\{y_n : n = 0,1,\cdots\}$ are two sequences of real numbers. In addition, we define $x(z) \sim y(z)$ when $z \to a$, which denotes that $\lim_{z\to a} \frac{x(z)}{y(z)} = 1$ where $x(z)$ and $y(z)$ are two functions. From the result of Section 3, we can get the below lemma.

Lemma 4.1. *For the queueing system with a retrial orbit and batch service, the stationary probability* $\pi_{n,i}$ *satisfies*

$$\pi_{n,i} \sim C_i h_i(n), \quad i = 0,1,\cdots,N,$$

where C_i *is a constant and independent of n, and* $h_i(n)$ *is shown in* (17).

The gist of this paper is to deduce the expression of C_i in Lemma 4.1. To achieve the goal, we introduce the vector $\boldsymbol{F}_n = \left(F_{n,0}, F_{n,1}, \cdots, F_{n,N}\right)$, where

$$F_{n,i} = n!\left(\frac{\alpha\bar{q}}{\lambda p}\right)^n \pi_{n,i}, \quad i = 0,1,2,\cdots,N, \tag{18}$$

and define its generating function $\boldsymbol{F}^*(z) = \left(F_0^*(z), F_1^*(z), \cdots, F_N^*(z)\right)$, where

$$F_i^*(z) = \sum_{n=0}^{\infty} F_{n,i} z^n, \quad i = 0,1,2,\cdots,N. \tag{19}$$

From Lemma 4.1, the following lemma can be obtained immediately.

Lemma 4.2. *The vector generating function* $\boldsymbol{F}^*(z)$ *is analytic on* $\{z \in \mathbb{C} : |z| < 1\}$, *that is,* $F_i^*(z)$ *is analytic on* $\{z \in \mathbb{C} : |z| < 1\}$ *for* $i = 0,1,\cdots,N$.

We duplicate the following lemma which is a special case of the Karamata Tauberian theorem on power series, and the lemma plays a very important role in the next section.

Lemma 4.3. (*Bingham et al.* [22]) *Suppose that* $a_n \ge 0$ *and the power series* $\sum_{n=0}^{\infty} a_n z^n$ *converges for* $z \in [0,1)$. *If* α *and r are real numbers with* $r \ge 0$, *then*

$$\sum_{k=0}^{n} a_k \sim \frac{\alpha n^r}{\Gamma(1+r)}, \text{ as } n \to \infty,$$

if and only if

$$\sum_{k=0}^{\infty} a_n z^n \sim \frac{\alpha}{(1-z)^r}, \text{ as } z \to 1^-.$$

Lemma 4.4. *If k is a non-negative integer which satisfies* $k > \frac{\theta}{\alpha\bar{q}} + N - i$, *then for* $i = 0,1,\cdots,N$, *we have*

$$0 < \liminf_{z\to 1^-} \frac{\frac{\mathrm{d}^k}{\mathrm{d}z^k} F_i^*(z)}{(1-z)^{\frac{\theta}{\alpha\bar{q}}+N-i-1-k}} \le \limsup_{z\to 1^-} \frac{\frac{\mathrm{d}^k}{\mathrm{d}z^k} F_i^*(z)}{(1-z)^{\frac{\theta}{\alpha\bar{q}}+N-i-1-k}} < \infty.$$

Proof. Based on Lemma 4.1 and (18), we can find two positive constants K_1 and K_2 which satisfy

$$K_1 n^{-\frac{\theta}{\alpha\bar{q}}-N+i} \le F_{n,i} \le K_2 n^{-\frac{\theta}{\alpha\bar{q}}-N+i}.$$

Then we can get another two positive constants K_1' and K_2' such that

$$K_1' n^k n^{-\frac{\theta}{\alpha\bar{q}}-N+i} \le \frac{(n+k)!}{n!} F_{n+k,i} \le K_2' n^k n^{-\frac{\theta}{\alpha\bar{q}}-N+i}. \tag{20}$$

From the definition of $F_i^*(z)$ in (19), we know

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} F_i^*(z) = \sum_{j=0}^{\infty} \frac{(j+k)!}{j!} F_{j+k,i} z^j. \tag{21}$$

Then we immediately obtain the following in the equation, for $z \in [0,1)$

$$K_1' \sum_{n=0}^{\infty} n^{k-\frac{\theta}{\alpha\bar{q}}-N+i} z^n \le \frac{\mathrm{d}^k}{\mathrm{d}z^k} F_i^*(z) \le K_2' \sum_{n=0}^{\infty} n^{k-\frac{\theta}{\alpha\bar{q}}-N+i} z^n.$$

According to Lemma 4.3, the proof is completed. □

Now the main conclusion about the expression of C_i in Lemma 4.1 is shown in the following theorem.

Theorem 4.1. *There exists a positive constant c such that*

$$\lim_{n\to\infty} \frac{\pi_{n,i}}{h_i(n)} = c\left(\frac{\alpha}{\theta}\right)^i, \quad i = 0,1,\cdots,N,$$

where $h_i(n)$ *is given by* (17) *and c is shown in* (27).

Proof. Based on the balance equation $\boldsymbol{\pi} Q = \mathbf{0}$, we can obtain the following differential equation after some calculations,

$$\boldsymbol{F}^*(z)\left(\sigma^2 z^2 \tilde{A} + \sigma z \tilde{B}_0 + \tilde{C}\right) + \frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{F}^*(z)\left(\sigma^2 z^3 \tilde{A} + \sigma z^2 \tilde{B}_1\right) = \boldsymbol{F}_0 \tilde{C}, \tag{22}$$

where $\sigma = \frac{\alpha\bar{q}}{\lambda p}$ and $\frac{\mathrm{d}}{\mathrm{d}z}\boldsymbol{F}^*(z)$ is interpreted component wise. Taking the *k*th derivative on both sides of (18) with respect to *z*, we have for $k \ge 2$,

$$\begin{aligned} &\sigma \frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}} \boldsymbol{F}^*(z)\left(\sigma z^3 \tilde{A} + z^2 \tilde{B}_1\right) + \frac{\mathrm{d}^k}{\mathrm{d}z^k}\boldsymbol{F}^*(z)\left[(3k+1)\sigma^2 z^2 \tilde{A} + \sigma z \tilde{B}_0 + 2k\sigma z \tilde{B}_1 + \tilde{C}\right] \\ &+ k\sigma \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}\boldsymbol{F}^*(z)\left[(3k-1)\sigma z \tilde{A} + \tilde{B}_0 + (k-1)\tilde{B}_1\right] + k(k-1)^2 \sigma^2 \frac{d^{k-2}}{dz^{k-2}} \boldsymbol{F}^*(z) \tilde{A} = 0. \end{aligned} \tag{23}$$

From now on, we take $k \equiv \left[\frac{\theta}{\alpha\bar{q}}\right] + 3$, where $[*]$ stands for the integer part.

Post multiplying (19) by $\boldsymbol{e}_{N+1}$, we obtain

$$-z^2(1-z)\frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}}F_N^*(z)+\left(k+1-\frac{\theta}{\alpha\bar{q}}\right)z^2\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_N^*(z)+\Phi^*(z)=0, \tag{24}$$

where

$$\begin{aligned}\Phi^*(z)=&\left[\left(2k+\frac{\theta}{\alpha\bar{q}}\right)z(z-1)+\frac{1}{\sigma}(1-z)\right]\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_N^*(z)\\&+\left(\frac{\lambda}{\alpha\bar{q}}z+\frac{1}{\bar{q}\sigma}\right)\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_{N-1}^*(z)\\&+\left[k(3k-1)z-\frac{k(\lambda p+\theta)}{\alpha\bar{q}}-k(k-1)\right]\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}F_N^*(z)\\&+\frac{\lambda k}{\alpha\bar{q}}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}F_{N-1}^*(z)+k(k-1)^2\frac{\mathrm{d}^{k-2}}{\mathrm{d}z^{k-2}}F_N^*(z).\end{aligned}$$

Noting that $\frac{\Phi^*(z)}{z^2(1-z)}$ has a removable singularity at $z=0$, we rewrite (24) as

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}}F_N^*(z)=\left(k+1-\frac{\theta}{\alpha\bar{q}}\right)\frac{1}{1-z}\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_N^*(z)+\frac{\Phi^*(z)}{z^2(1-z)},\quad |z|<1. \tag{25}$$

Solving (25) for $\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_N^*(z)$, we have, for $|z|<1$,

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_N^*(z)=(1-z)^{\frac{\theta}{\alpha\bar{q}}-k-1}\left(F_N^{*(k)}(0)+\int_0^z\frac{\Phi^*(t)}{t^2}(1-t)^{k-\frac{\theta}{\alpha\bar{q}}}\,\mathrm{d}t\right),$$

where $F_N^{*(k)}(0)=\left.\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_N^*(z)\right|_{z=0}$. Based on Lemma 4.4, we find that $\Phi^*(t)(1-t)^{k-\frac{\theta}{\alpha\bar{q}}}$ is bounded when $t\in(0,1)$. When $z\to1^-$, we can obtain the following formula from (25)

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_N^*(z)\sim c\left(\frac{\alpha}{\theta}\right)^N\Gamma\left(k+1-\frac{\theta}{\alpha\bar{q}}\right)(1-z)^{\frac{\theta}{\alpha\bar{q}}-k-1}, \tag{26}$$

where

$$c=\left(\frac{\theta}{\alpha}\right)^N\frac{F_N^{*(k)}(0)+\int_0^1\frac{\Phi^*(t)}{t^2}(1-t)^{k-\frac{\theta}{\alpha\bar{q}}}\,\mathrm{d}t}{\Gamma\left(k+1-\frac{\theta}{\alpha\bar{q}}\right)}. \tag{27}$$

Recalling (21) and applying Lemma 4.3 to (26), we can obtain that

$$\sum_{j=0}^n\frac{(j+k)!}{j!}F_{j+k,N}\sim c\left(\frac{\alpha}{\theta}\right)^N\frac{1}{k+1-\frac{\theta}{\alpha\bar{q}}}n^{k+1-\frac{\theta}{\alpha\bar{q}}},\quad n\to\infty. \tag{28}$$

Then we compare the coefficients of z^n for both sides of (25). It is easy to find that the coefficient of z^n on the left side is

$$\frac{(n+k+1)!}{n!}F_{n+k+1,N},$$

and the coefficient of z^n, the first term on the right side, is

$$\left(k+1-\frac{\theta}{\alpha\bar{q}}\right)\sum_{j=0}^{n}\frac{(j+k)!}{j!}F_{j+k,N}.$$

By calculating the second term on the right side of (25), we have

$$\begin{aligned}\frac{\Phi^*(z)}{1-z}=&-\left(2k+\frac{\theta}{\alpha\bar{q}}\right)z\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_N^*(z)+\frac{1}{\sigma}\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_N^*(z)\\&+\frac{\lambda}{\alpha\bar{q}}\frac{z}{1-z}\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_{N-1}^*(z)+\frac{1}{\sigma\bar{q}}\frac{1}{1-z}\frac{\mathrm{d}^k}{\mathrm{d}z^k}F_{N-1}^*(z)\\&+k(3k-1)\frac{z}{1-z}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}F_N^*(z)-k\left(\frac{\lambda p+\theta}{\alpha\bar{q}}+k-1\right)\frac{1}{1-z}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}F_N^*(z)\\&+\frac{\lambda k}{\alpha\bar{q}}\frac{1}{1-z}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}F_{N-1}^*(z)+k(k-1)^2\frac{1}{1-z}\frac{\mathrm{d}^{k-2}}{\mathrm{d}z^{k-2}}F_N^*(z).\end{aligned}$$

Let Φ_n $(n=0,1,2,\cdots)$ be the coefficients of the power series expansion of $\Phi^*(z)$ about $z=0$. By comparing the coefficients of z^n on both sides of the above equation, we get

$$\begin{aligned}\sum_{j=0}^{n}\Phi_j=&-\left(2k+\frac{\theta}{\alpha\bar{q}}\right)\frac{(n+k-1)!}{(n-1)!}F_{n+k-1,N}+\frac{1}{\sigma}\frac{(n+k)!}{n!}F_{n+k,N}\\&+\frac{\lambda}{\alpha\bar{q}}\sum_{j=0}^{n-1}\frac{(j+k)!}{j!}F_{j+k,N-1}+\frac{1}{\sigma\bar{q}}\sum_{j=0}^{n}\frac{(j+k)!}{j!}F_{j+k,N-1}\\&+k(3k-1)\sum_{j=0}^{n-1}\frac{(j+k-1)!}{j!}F_{j+k-1,N}\\&-k\left(\frac{\lambda p+\theta}{\alpha\bar{q}}+k-1\right)\sum_{j=0}^{n}\frac{(j+k-1)!}{j!}F_{j+k-1,N}\\&+\frac{\lambda k}{\alpha\bar{q}}\sum_{j=0}^{n}\frac{(j+k-1)!}{j!}F_{j+k-1,N-1}+k(k-1)^2\sum_{j=0}^{n}\frac{(j+k-2)!}{j!}F_{j+k-2,N}.\end{aligned}\tag{29}$$

Then for the coefficients of z^n on both sides of (25), we obtain the following equation

$$\frac{(n+k+1)!}{n!}F_{n+k+1,N}=\left(k+1-\frac{\theta}{\alpha\bar{q}}\right)\sum_{j=0}^{n}\frac{(j+k)!}{j!}F_{j+k,N}+\sum_{j=0}^{n}\Phi_{j+2},\quad n=0,1,2,\cdots.\tag{30}$$

From (20) and (28), (29) yields

$$\lim_{n\to\infty}\frac{\sum_{j=0}^{n}\Phi_j}{n^{k-\frac{\theta}{\alpha\bar{q}}+1}}<\infty.$$

Dividing both sides of (30) by $n^{k-\frac{\theta}{\alpha\bar{q}}+1}$ and taking $n\to\infty$, we have

$$F_{n,N}\sim c\left(\frac{\alpha}{\theta}\right)^N n^{-\frac{\theta}{\alpha\bar{q}}},$$

which leads to

$$\pi_{n,N}\sim c\left(\frac{\alpha}{\theta}\right)^N h_N(n)$$

from (18). This result indicates that the theorem holds for $i = N$.

From the balance equation $\boldsymbol{\pi} Q = \mathbf{0}$, for $n = 0,1,\cdots$ and $i = 0,1,2,\cdots,N-2$, we get

$$\begin{cases} -\left(\lambda+(n+1)\alpha\right)\pi_{n+1,0}+\theta\pi_{n+1,1}+\sum_{j=1}^{N-1}\eta_j\pi_{n+1,j}=0, & (31)\\ \lambda\pi_{n+1,i}-\left[\kappa_{i+1}+(n+1)\alpha\right]\pi_{n+1,i+1}+\theta\pi_{n+1,i+2}+(n+2)\alpha\pi_{n+2,i}=0, & (32)\\ \lambda p\pi_{n,N}+\lambda\pi_{n+1,N-1}+\xi_{n+1}\pi_{n+1,N}+(n+2)\alpha\left(\pi_{n+2,N-1}+\bar{q}\pi_{n+2,N}\right)=0, & (33)\end{cases}$$

where $\xi_{n+1} = -\left(\lambda p+\theta+(n+1)\alpha\bar{q}\right)$. Based on Lemma 4.1, we find, for $i = 0,1,2,\cdots,N$,

$$\lim_{n\to\infty}\frac{\pi_{n,i}}{h_i(n)}<\infty.$$

Dividing both sides of (31)-(33) by $h_{i+1}(n)$ and taking $n\to\infty$, we get

$$n\alpha\pi_{n,i+1}\sim\theta\pi_{n,i},\quad i=0,1,\cdots,N-1.$$

The proof is finished. □

5. Concluding Remarks

In this paper, we discussed a queueing system with an orbit and batch service which depends on the batch size. We not only got the decay function of the stationary distribution, but also derived a more accurate tail asymptotic form. Using the method described in this article, we may analyze the exact tail asymptotic of other similar queueing systems.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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