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A New Uniformly Ultimate Boundedness Criterion for Discrete-Time Nonlinear Systems

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Abstract

A new type criterion of globally uniformly ultimate boundedness for discrete-time nonlinear systems is introduced. In classical Lyapunov theory about globally uniformly ultimate boundedness, Lyapunov function is assumed to be positive definite and its difference at the every latter moment and the former moment is negative definite. In this paper the condition of difference of Lyapunov function is relaxed. Under the relaxed condition, the result of this paper can be considered as the extension of the classical Lyapunov theory about uniformly ultimate boundedness.

Keywords: Uniformly Ultimate Boundedness, Lyapunov Function, Discrete-Time Nonlinear Systems

1. Introduction

Discrete-time systems of difference equations have attracted considerable attention. A remarkable book [1] by Agarwal makes the general survey on theory of difference equation and introduces a detailed account of the application of difference equations. Stability analysis is one of the main issues in the area of control systems theory.The classical Lyapunov theory [2] is an important tool to discuss the stability and stabilization problems of dynamical systems. A candidate Lyapunov function is required to satisfy positive definite, and its difference be negative definite or semi-negative definite at the every latter moment and the former moment [3-4]. For discrete-time nonlinear systems, the relation of smooth Lyapunov function and asymptotical stability is presented in paper [5]. In paper [6], new results on the existence of Lyapunov function are presented for discrete-time systems described by difference inclusions. Some new characterizations of uniform global asymptotic stability for nonlinear timevarying discrete-time systems with or without an output-dominant perturbation are proposed on the basis of the detectability for the reduced limiting systems associated with the original system in paper [7]. A new asymptotic stability criterion for nonlinear time-varying differential equations is demonstrated in paper [8] by Aeyels and Peuteman. The Lyapunov function need not be differentiable, and not even be Lipschitz continuous. Under the relaxed condition, a new asymptotic stability criterion is introduced.

Uniform boundness and uniformly ultimate boundedness is an indispensable part of stability problems [1,2]. In paper [9], Aeyels, Peuteman and Sepulchre transform the problem of uniform boundedness and uniform ultimate boundedness for nonautonomous continuous systems to time-invariant frozen systems and introduced some important results. Bu and Mu [10] extend those results and present Lyapunov theorems of uniform boundedness and uniform ultimate boundedness for nonautonomous homogeneous systems. Paper [11] by Cheng, Mu and Ding discusses the problem of uniformly ultimate boundedness of nonautonomous nonlinear systems with discontinuous right-hand sides and gives some results based on differential inclusions and Filippov solutions. Under arbitrary switching laws, a continuous state feedback control [12] scheme is proposed in order to guarantee uniformly ultimate boundedness of every system response within an arbitrary small neighborhood of the zero state.

In the research area about uniformly ultimate boundedness, the condition of Lyapunov function usually considered is differentiable or Lipschitz continuous and regular. In this paper, based on [8] given by Aeyels and Peuteman, we relax the assumption condition of Lyapunov function. We don’t suppose the difference of Lyapunov function at the every latter moment and the former moment is negative definite but suppose there exists a finite integer such that Lyapunov function satisfies certain condition. Our object is to provide a new uniformly ultimate boundedness criterion for discrete-time nonlinear systems.

Without loss of generality, R^n denotes the n-dimensional Euclidean space, the notation $\|\cdot\|$ is used to denote the Euclidean 2-norm of a vector. Z denotes the set of integers. Continuous function $\alpha(\cdot)$ is said to be a function of class K if, it is strictly increasing and $\alpha(0)=0$. A closed ball B_τ is denoted by $B_\tau = \left\{ x \middle| \|x\| \le \tau \right\}$.

The rest of the paper is organized as follows. Mathematical preliminary is stated in Section 2. A new type criterion of uniformly ultimate boundedness for nonlinear discrete-time systems is proposed in Section 3. A brief conclusion is provided to summarize the paper in Section 4. Finally, acknowledgements are given in the final section.

2. Mathematical Preliminary of the Problem

Consider the systems

$$x(k+1) = f(x(k),k) \tag{1}$$

where $x(k) \in R^n$ is the state vector at time instant k, $k \in Z$. We assume that $f: R^n \times Z \to R^n$ satisfies $f(0,k)=0$ for all $k \in Z$ and is globally Lipschitz continuous. Without loss of generality, let L be the Lipschitz constant. Thus the existence and uniqueness of the solution of system (1) is satisfied. $x(k)$ denotes the solution with the initial value $x(k_0)=x_0$.

Firstly the definitions of globally uniform boundedness and uniformly ultimate boundedness for discrete-time nonlinear systems (1) arepresented.

Definition 2.1. The origin of system (1) is said to be globally uniformly bounded if, for any positive constant $a>0$, there is $b=b(a)>0$ (independent of k_0), such that when $\|x(k_0)\| \le a$, there holds

$$\|x(k)\| \le b, \forall k \ge k_0 .$$

Definition 2.2. The origin of system (1) is said to be globally uniformly ultimately bounded if, there exist positive constants r, for all $\sigma>0$, there is $K=K(\sigma,r)\ge 0$ (independent of k_0), such that when $\|x(k_0)\| \le \sigma$, there holds

$$\|x(k)\| \le r, \forall k \ge k_0 + K(\sigma,r) .$$

Lemma 2.3. ([5]) For all $\mu>0$, and choose the closed ball $B_\mu \subset R^n$. Then for any finite integer $m>0$, there exists $\mu'>0$ such that for all $(x_0,k_0) \in B_{\mu'} \times Z$, there holds

$$\|x(k)\| \le \mu, \forall k \in [k_0, k_0+m] .$$

3. Main Result

A new type globally uniformly ultimate boundedness criterion for nonlinear discrete-time systems is proposed in this section.

Theorem 3.1. There exists a Lyapunov function $W: R^n \times Z \to R$ which satisfies:

1) there exist two functions α_1, α_2 of class K, such that

$$\alpha_1\left(\|x(k)\|\right) \le W\left(x(k),k\right) \le \alpha_2\left(\|x(k)\|\right); \tag{2}$$

2) for all $\mu>0$, choose $\mu' = \mu e^{-Lm}$, there exists $0<\tau<\alpha_2^{-1}\left(\alpha_1(\mu')\right)$, while for all $\|x\|\ge\tau$, there exists a function α_3 of class K, and a finite integer $m>0$, such that

$$W\left(x(k+m),k+m\right) - W\left(x(k),k\right) \le -\alpha_3\left(\|x(k)\|\right). \tag{3}$$

Then the origin of system (1) is globally uniformly ultimate bounded.

Proof: Choose arbitrary initial value $(x_0,k_0) \in B_{\mu'} \times Z$, there are only two cases:

(I) $W(x_0,k_0) \le \alpha_2(\tau)$; (II) $W(x_0,k_0) > \alpha_2(\tau)$.

Case I: When (I) holds, then we have

$$W\left(x(k_0+m),k_0+m\right) < W\left(x(k_0),k_0\right) \le \alpha_2(\tau);$$

and

$$W\left(x(k_0+2m),k_0+2m\right) < W\left(x(k_0),k_0\right) \le \alpha_2(\tau).$$

By iterative approach, for all $l \in Z_{\ge 0}$, there holds

$$W\left(x(k_0+lm),k_0+lm\right) < W\left(x(k_0),k_0\right) \le \alpha_2(\tau).$$

From (2), we have

$$\|x(k_0+lm)\| \le \alpha_1^{-1} \circ \alpha_2\left(\alpha_2^{-1}\left(\alpha_1\left(\mu'\right)\right)\right) = \mu'.$$

Hence for all $k \ge k_0$, there exists a constant $k \in Z_{\ge 0}$, such that $k-k_0-Km=m$, and

$$\|x(k)\| = \left\|x\left(k_0+(K+1)m\right)\right\| \le \mu'.$$

By Lemma 2.1, for all $k \ge k_0$, there has

$$\|x(k)\| \le \mu' e^{Lm} = \mu .$$

Case II: When (II) holds, Let

$$K^* = \sup\left\{ k \middle| W\left(x\left(k_0+sm\right),k_0+sm\right) > \alpha_2(\tau) \right\}, \tag{4}$$

where $0 \le s \le k$, $k \in Z_{\ge 0}$.

We claim $K^* < +\infty$.

Contradiction If not, there must hold $K^* = +\infty$, and for all $k \in Z_{\ge 0}$, there has

$$W\left(x(k_0+km),k_0+km\right) > \alpha_2(\tau).$$

Then

$$\alpha_2(\tau) < W\left(x(k_0+km),k_0+km\right) < \alpha_2\left(\|x(k_0+km)\|\right),$$

which implies

$$\|x(k_0+km)\| > \tau ; k \in Z_{\ge 0} .$$

By (4) and iteration, we can get the following inequalities

$$\begin{aligned}&W\left(x(k_0+(k+1)m), k_0+(k+1)m\right)\\ &-W\left(x(k_0+km), k_0+km\right)\\ &\le -\alpha_3\left(\|x(k_0+km)\|\right) \le -\alpha_3(\tau);\end{aligned}$$

$$\begin{aligned}&W\left(x(k_0+km), k_0+km\right)\\ &-W\left(x(k_0+(k-1)m), k_0+(k-1)m\right)\\ &\le -\alpha_3\left(\|x(k_0+(k-1)m)\|\right) \le -\alpha_3(\tau);\end{aligned}$$

$$\begin{aligned}&W\left(x(k_0+Tm), k_0+Tm\right) - W\left(x(k_0), k_0\right)\\ &< -\alpha_3(k_0) < -\alpha_3(\tau),\end{aligned}$$

then

$$\begin{aligned}&W\left(x\left(k_0+(k+1)m\right), k_0+(k+1)m\right)\\ &< W\left(x(k_0), k_0\right) - (k+1)\alpha_3(\tau).\end{aligned}$$

When $K \to +\infty$, clearly $W\left(x\left(k_0+(k+1)m\right), k_0+(k+1)m\right) \to -\infty$, which contradicts to the positive definite property of $W\left(x(k), k\right)$.

Then there exists an integer $K^* \in Z_{\ge 0}$, when $k < K^*$, we have $W\left(x(k_0+km), k_0+km\right) > \alpha_2(\tau)$; when $k < K^*$, we have

$$W\left(x(k_0+km), k_0+km\right) \le \alpha_2(\tau) .$$

Consequently

$$\|x(k_0+km)\| \le \alpha_1^{-1} \circ \alpha_2(\alpha_2^{-1}(\alpha_1(\mu'))) = \mu' .$$

Furthermore, we can estimate K^*. By iteration, there has

$$\begin{aligned}&W\left(x\left(k_0+(K^*-1)m\right), k_0+(K^*-1)m\right)\\ &< W\left(x(k_0), k_0\right) - (K^*-1)\alpha_3(\tau)\end{aligned}$$

i.e.

$$\begin{aligned}&K^*\\ &< \frac{W\left(x(k_0), k_0\right) - W\left(x\left(k_0+(K^*-1)m\right), k_0+(K^*-1)m\right)}{\alpha_3(\tau)} + 1\\ &< \frac{\alpha_2(\sigma) - \alpha_2(\tau))}{\alpha_3(\tau)} + 1.\end{aligned}$$

Then choose

$$K^*(\sigma, \tau) = \frac{\alpha_2(\sigma) - \alpha_2(\tau))}{\alpha_3(\tau)} + 1 = K^*(\sigma, \mu') = K^*(\sigma, \mu)$$

where $K^*(\sigma, \mu)$ is independent of t_0.

Therefore

$$\|x(k_0+kT)\| \le \alpha_1^{-1} \circ \alpha_2(\alpha_2^{-1}(\alpha_1(\mu'))) = \mu' .$$

So in view of Lemma 2, for all $k > k_0 + K^*(\sigma, \mu)$,

$$\|x(k)\| \le \mu' e^{Lm} = \mu .$$

The proof is completed.

4. Conclusions

We conclude with a brief discussion. In this paper, an extensive Lyapunov theorem of uniform ultimately boundedness is presented. In the classical Lyapunov theory about uniform ultimately boundedness, the difference of Lypunov function at the every latter moment and the former moment is negative definite. Here only need to exist an integer m such that the condition 2) of Theorem 3.1 is satisfied. When $m = 1$, the condition 2) becomes the condition in classical Lyapunov theory. Thus theorem 3.1 is less restrictive than that in the classical Lyapunov theory.

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