Generating Totally Positive Toeplitz Matrix from an Upper Bidiagonal Matrix

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Abstract

In this paper, we construct one of the forms of totally positive Toeplitz matrices from upper or lower bidiagonal totally nonnegative matrix. In addition, some properties related to this matrix involving its factorization are presented.

Keywords

Totally Positive Matrix, Totally Nonnegative Matrix, Toeplitz Matrix, *LU* Factorization

1. Introduction

Total positive matrices arise in many areas in mathematics, and there has been considerable interest lately in the study of these matrices. For background information see the most important survey in this field by T. Ando [1]. See also [2].

A matrix A is said to be totally positive, if every square submatrix has positive minors and A is said to be totally nonnegative, and if every square submatrix has nonnegative minors. While it is well known that many of the nontrivial examples of totally positive matrices are obtained by restricting certain kernels to appropriate finite subsets of R (see, for example, Ando ([1], p. 212) or Pinkus ([3], p. 2). For Toeplitz matrices, that is, matrices of the form $T = \left(t_{i-j} \right)_{i,j=1}^{n}$, a complete characterization of the total positivity, in terms of certain entire functions, has been studied in a series of references by Ando [1], Pinkus [3] and S.M. Fallat, C.R. Johnson [4].

Expressing a matrix as a product of lower triangle matrix L and an upper triangle matrix U is called a *LU* factorization. Such factorization is typically obtained by reducing a matrix to an upper triangular matrix from via row operation, that is, Gaussian elimination.

The primary purpose of this paper is to provide a new totally positive matrix generated from a totally nonneg-

ative one and to construct its factorization.

The organization of our paper is as follows. In Section 2, we introduce our notation and give some auxiliary results which we use in the subsequent sections. In Section 3, we recall from [3] the Toeplitz matrices specified for the case $t_k = \binom{n}{k}$, on which our proofs heavily rely. In Section 4, we present the proofs of our main results. In last section, we present the factorization of this resulted matrix.

2. Notation and Auxiliary Results

2.1. Notations

In this subsection we introduce the notation that will be used in developing the paper. For k, n we denote by $Q_{k,n}$ the set of all strictly increasing sequences of k integers chosen from $\{1, 2, \cdots, n\}$. For $\alpha = \{\alpha_1, \cdots, \alpha_k\}$, $\beta = \{\beta_1, \cdots, \beta_k\} \in Q_{k,n}$, we denote by $A[\alpha|\beta]$ the $k \times k$ submatrix of A contained in the rows indexed by $\alpha_1, \cdots, \alpha_k$ and columns indexed by $\beta_1, \cdots, \beta_k$. A matrix A is called totally positive (abbreviated *TP* henceforth) and totally nonnegative (abbreviated TN) if $\det A[\alpha|\beta] > 0$ and $\det A[\alpha|\beta] \geq 0$, respectively, for all α, $\beta \in Q_{k,n}$, $k = 1, 2, \cdots, n$. If a totally nonnegative matrix is also nonsingular, we write NsTN.

Definition 2.1.1 [3]

A square lower (upper) triangular matrix A is called lower (upper) triangular positive matrix, denoted L*TP* (U*TP*), if for all $k = 1, \cdots, n$ and for $\alpha = \{\alpha_1, \cdots, \alpha_k\}$, $\beta = \{\beta_1, \cdots, \beta_k\} \in Q_{k,n}$ with the property that $\alpha_s \geq \beta_s$ ($\beta_s \geq \alpha_s$) for $s = 1, \cdots, k$, then $\det A[\alpha|\beta] > 0$.

Let I be the square identity matrix of order n, and for $1 \leq i, j \leq n$, we let E_{ij} be the square standard basis matrix whose only nonzero entry is 1 that occurs in the (i, j) position.

A tridiagonal matrix that is also upper (lower) triangular is called an upper (lower) bidiagonal matrix. Statements referring to just triangular or bidiagonal matrices without the adjectives "upper" or "lower" may be applied to either case.

2.2. Auxiliary Results

We use the following classic formula known as Cauchy-Binet formula and stated in the theorem below.

Theorem 2.2.1 (Cauchy-Binet formula) ([4], p. 27). Let A be an $n \times m$ matrix and B be an $m \times p$ matrix then for each pair of indexed sets $\alpha \subseteq \{1, \cdots, n\}$ and $\beta \subseteq \{1, \cdots, p\}$ of cardinality k, where $1 \leq k \leq \min\{n, m, p\}$, we have

$$\det AB[\alpha, \beta] = \sum_{\eta, |\eta| = k} \det A[\alpha, \eta] \cdot \det B[\eta, \beta]$$

The following remarkable result is one of the most important and useful results in the study of TN matrices. This result first appeared in [5] see also [1] for another proof of this fact.

Theorem 2.2.2. Let $A = (a_{ij})_{i,j=1}^n$ be a square matrix of order n. Then A is NsTN if and only if A has an *LU* factorization, such that both L and U are NsTN square matrices.

Using this theorem and Cauchy-Binet formula we have the following corollary.

Corollary 2.2.3 [6]. Let $A = (a_{ij})_{i,j=1}^n$ be a square matrix of order n. Then A is *TP* if and only if A has an *LU* factorization, such that both L and U are *TP* square matrices.

We have the following theorem to prove both L and U are totally positive.

Theorem 2.2.4. Let $U = (u_{ij})_{i,j=1}^n$ be an upper triangular square matrix of order n satisfying

$$U[1, \cdots, k | j+1, \cdots, j+k] > 0 \text{ for } j = 0, 1, \cdots, n-k, \ k = 1, \cdots, n$$

Then U is U*TP* (upper totally positive). Similarly, if $L = (l_{ij})_{i,j=1}^n$ is an lower triangular square matrix of order n satisfying $L[i+1, \cdots, i+k | 1, \cdots, k] > 0$ for $j = 0, 1, \cdots, n-k$, $k = 1, \cdots, n$. Then L is L*TP* (lower totally positive).

In the sequel we will make use the the following lemma, see, e.g. [7].

Lemma 2.2.5 (Sylvester Identity)

Partition square matrix T of order n, $n > 2$, as:

$$T = \begin{bmatrix} c_1 & T_{12} & c_2 \\ T_{21} & T_{22} & T_{23} \\ c_3 & T_{32} & c_4 \end{bmatrix},$$

where T_{22} square matrix of order $n-2$ and c_1, c_2, c_3 and c_4 are scalars. Define the submatrices

$$T_1 = \begin{bmatrix} c_1 & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad T_2 = \begin{bmatrix} T_{12} & c_2 \\ T_{22} & T_{23} \end{bmatrix}, \quad T_3 = \begin{bmatrix} T_{21} & T_{22} \\ c_3 & T_{32} \end{bmatrix}, \quad T_4 = \begin{bmatrix} T_{22} & T_{23} \\ T_{32} & c_4 \end{bmatrix}$$

Then if T_{22} is non singular

$$\det T = \frac{\det T_1 \det T_4 - \det T_2 \det T_3}{\det T_{22}}$$

3. Toeplitz Matrices

Assuming we are given a finite sequence $t_{-n}, t_{-n+1}, \cdots, t_{-1}, t_0, t_1, \cdots, t_n$ of distinct real numbers, the associated Toeplitz matrix is defined by $T = \left(t_{j-i} \right)_{i,j=1}^{n}$ or $T = \left(t_{i-j} \right)_{i,j=1}^{n}$. If we are given a one-sided finite sequence $t_0, t_1, \cdots, t_n$, then we understand this to mean that $t_{-k} = 0, k = 1, 2, \cdots, n$ in the above definition. Sequences that give rise to totally positive Toeplitz matrices have been totally characterized in terms of their generating functions, *i.e.* representations of $\sum_{k=-n}^{n} t_k x^k$.

In our case, the normalization $t_0 = 1$, the sequence $t_0, t_1, \cdots, t_n$ gives rise to a totally positive Toeplitz matrix $T = \left(t_{j-i} \right)_{i,j=1}^{n}$ if and only if $\sum_{k=0}^{n} t_k x^k$ has the form

$$e^{\lambda x} \frac{\prod_{s=1}^{n} \left(1 + i_s x \right)}{\prod_{s=0}^{n} \left(1 - j_s x \right)}$$

where $\lambda \geq 0, i_s \geq 0, j_s \geq 0$.

Now consider the polynomial $f(x) = (1+x)^n$, the upper triangular Toeplitz matrix

$$U = \begin{bmatrix} \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-1} \\ 0 & \binom{n}{0} & \binom{n}{1} & \ddots & \binom{n}{n-2} \\ 0 & 0 & \binom{n}{0} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \binom{n}{1} \\ 0 & 0 & 0 & 0 & \binom{n}{0} \end{bmatrix}$$

is *TP*.

4. Generating New Form of Toeplitz Matrix

4.1. Main Result

Now we formalize the structure of our result by the following theorem.

Theorem 4.1.1. Assume that we are given the sequence $x_1, \cdots, x_{n-1}$ of $(n-1)$ distinct positive real numbers. Define the upper bidiagonal matrix $P = (p_{ij})_{i,j=1}^{n}$ by

$$p_{ij} = \begin{cases} x_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

That is the sequence $x_1, \cdots, x_{n-1}$ lies on the superdiagonal. Then the matrix T defined as

$$T = f(P) = \sum_{r=0}^{n}\sum_{k=0}^{n}\binom{n}{r}\binom{n}{k}P^r \left(P^T\right)^k$$

is *TP*.

Proof

To prove this result we must note that

$$T = f(P) = \sum_{r=0}^{n}\sum_{k=0}^{n}\binom{n}{r}\binom{n}{k}P^r \left(P^T\right)^k = \left(\sum_{r=0}^{n}\binom{n}{r}P^r\right) \times \left(\sum_{k=0}^{n}\binom{n}{k}\left(P^T\right)^k\right) = UL$$

where $U = \left(\sum_{r=0}^{n}\binom{n}{r}P^r\right)$ is upper triangular matrix and $L = \left(\sum_{k=0}^{n}\left(P^T\right)^k\right)$ is lower triangular matrix. By corollary 2.2.3 A is *TP* if both U and L are TP.

So, want to prove $U = \left(\sum_{r=0}^{n}\binom{n}{r}P^r\right)$ is upper TP.

$$U = \begin{bmatrix} \binom{n}{0} & \binom{n}{1}x_1 & \binom{n}{2}x_1 x_2 & \cdots & \binom{n}{n-1}\prod_{i=1}^{n-1}x_i \\ 0 & \binom{n}{0} & \binom{n}{1}x_2 & \ddots & \binom{n}{n-2}\prod_{i=1}^{n-2}x_i \\ 0 & 0 & \binom{n}{0} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \binom{n}{1}x_{n-1} \\ 0 & 0 & 0 & 0 & \binom{n}{0} \end{bmatrix}$$

By **Theorem 2.2.4** U is *TP* if

$$U\left[1,\cdots,k \mid j+1,\cdots,j+k\right] = q\begin{vmatrix} \binom{n}{j} & \binom{n}{j+1} & \cdots & \binom{n}{j+k-1} \\ \binom{n}{j-1} & \binom{n}{j} & \cdots & \binom{n}{j+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{j-k+1} & \binom{n}{j-k} & \cdots & \binom{n}{j} \end{vmatrix} > 0$$

where $q = \left(\prod_{k=1}^{\frac{n}{2}+1} (x_k)^k \right) \left(\prod_{k=1}^{\frac{n}{2}} (x_{n-k})^k \right)$ which is positive and

$$
\begin{vmatrix}
\binom{n}{j} & \binom{n}{j+1} & \cdots & \binom{n}{j+k-1} \\
\binom{n}{j-1} & \binom{n}{j} & \cdots & \binom{n}{j+k-2} \\
\vdots & \vdots & \ddots & \vdots \\
\binom{n}{j-k+1} & \binom{n}{j-k} & \cdots & \binom{n}{j}
\end{vmatrix} > 0
$$

Since its submatrix of Toeplitz matrix.

Illustrative Example

Let we have the following sequence of distinct positive real numbers 1, 4, 3.
Define the matrix A as:

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Then the matrix function

$$
f(A) = \left(\binom{4}{0} A^0 + \binom{4}{1} A^1 + \binom{4}{2} A^2 + \binom{4}{3} A^3 + \binom{4}{4} A^4 \right)
$$

$$
\times \left(\binom{4}{0} (A^T)^0 + \binom{4}{1} (A^T)^1 + \binom{4}{2} (A^T)^2 + \binom{4}{3} (A^T)^3 + \binom{4}{4} (A^T)^4 \right)
$$

$$
= \left(I + 4A + 6A^2 + 4A^3 + A^4 \right) \times \left(I + 4A^T + 6(A^T)^2 + 4(A^T)^3 + (A^T)^4 \right)
$$

$$
= \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + 4 \times \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 6 \times \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + 4 \times \begin{bmatrix} 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)
$$

$$
\times \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + 4 \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} + 6 \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \end{bmatrix} + 4 \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)
$$

$$
= \begin{bmatrix} 1 & 4 & 24 & 48 \\ 0 & 1 & 16 & 72 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 24 & 16 & 1 & 0 \\ 48 & 72 & 12 & 1 \end{bmatrix} = \begin{bmatrix} 2897 & 3844 & 600 & 48 \\ 3844 & 5441 & 880 & 72 \\ 600 & 880 & 145 & 12 \\ 48 & 72 & 12 & 1 \end{bmatrix}
$$

is *TP*.

4.2. Properties

1) Note that $\det T = 1$ since $T = \left(\sum_{r=0}^{n} \binom{n}{r} P^r \right) \times \left(\sum_{k=0}^{n} \binom{n}{k} \left(P^{\mathrm{T}} \right)^k \right) = UL$ and

$$\det T = \det UL = \det U \, \det L = 1 \times 1 = 1$$

Using this property we prove the following lemma

Lemma 4.2.1. The matrix T, as defined above has the following property

$$\det T_{22} = \det T_1 \, \det T_4 - \det T_2 \, \det T_3$$

where T_1, T_2, T_3, T_4 and T_{22} are defined in **Lemma 2.2.5**.

Proof

The statement follows by **Lemma 2.2.5** and the idea of $\det T = 1$.

2) Let P denote the square matrix of order n permutation matrix by the permutation $p_{ij} = p_{n-i+1, n-j+1}$, $1 \leq i, j \leq n$, and suppose T is a square TP Toeplitz matrix. Then $PTP = LU$ is TP too (see [7]). Moreover, $ST^{-1}S$ is TP, where S is diagonal matrix with diagonal entries alternately 1 and -1.

3) The Hadamrd product of two TP toeplitz matrices is TP matrix too, that is if we are given two square TP matrices $T = \begin{bmatrix} t_{ij} \end{bmatrix}$ and $W = \begin{bmatrix} w_{ij} \end{bmatrix}$ of order n. Then the Hadamard product $Y = \begin{bmatrix} y_{ij} \end{bmatrix} = \begin{bmatrix} t_{ij} \cdot w_{ij} \end{bmatrix}$ is TP.

5. Factorization

5.1. Construct New Factorization

Our aim is to write the new TP Toeplitz matrix T as a product of elementary matrices of a special form. For any $k \in \{2, \cdots, n\}$, we let $E_k^r(\delta) = \begin{bmatrix} e_{ij} \end{bmatrix}$ to be the elementary lower matrix whose entries are defined by

$$e_{ij} = \begin{cases} 1 & \text{if } i = j \\ \delta & \text{if } i = k, j = k - r \\ 0 & \text{otherwise} \end{cases}$$

Note that $E_k^r(\delta)$ can be written as $E_k^r(\delta) = I + \delta E_{k, k-r}$, where I is square identity matrix of order n and $E_{i,j}$ is square matrix of order n whose non-zero entry is a 1 in the $(i, j)^{th}$ position n. Also, notice that $E_k^r(-\delta) = E_k^r(\delta)^{-1}$.

We use the elementary matrices $E_k^r(\delta)$ to reduce Lower diagonal matrix to identity matrix.

For example, we can consider the following 4×4 Lower diagonal matrix L

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 4 & 6 & 4 & 1 \end{bmatrix}$$

It can be factorized as

$$L = E_2^1(4) E_3^2(6) E_3^1(4) E_4^3(4) E_4^2(6) E_4^1(4) I$$

$$= \left(E_2^1(1) \right)^4 \left(E_3^2(1) \right)^6 \left(E_3^1(1) \right)^4 \left(E_4^3(1) \right)^4 \left(E_4^2(1) \right)^6 \left(E_4^1(1) \right)^4 I$$

5.2. General Characterization

We begin a definition and a result that characterize the TP Toeplitz matrix T in terms of the elementary matrices $E_k^l(1)$.

Theorem 5.2.1. Any square Toeplitz matrix of oreder n, $T = \begin{bmatrix} t_{ij} \end{bmatrix}$ can be written as

$$T = UL = \begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix}$$

$$= \left(\left(E_n^1(1) \right)^{u_{(n-1)n}} \right)^{\mathrm{T}} \left(\left(E_n^2(1) \right)^{u_{(n-2)n}} \right)^{\mathrm{T}} \cdots \left(\left(E_4^3(1) \right)^{u_{14}} \right)^{\mathrm{T}} \left(\left(E_3^1(1) \right)^{u_{23}} \right)^{\mathrm{T}} \left(\left(E_3^2(1) \right)^{u_{13}} \right)^{\mathrm{T}} \left(\left(E_2^1(1) \right)^{u_{12}} \right)^{\mathrm{T}} I$$

$$\cdot \left(E_2^1(1) \right)^{l_{21}} \left(E_3^2(1) \right)^{l_{31}} \left(E_3^1(1) \right)^{l_{32}} \left(E_4^3(1) \right)^{l_{41}} \cdots \left(E_n^2(1) \right)^{l_{n(n-2)}} \left(E_n^1(1) \right)^{l_{n(n-1)}}$$

That is, $T = \prod_{k=2}^{n} \prod_{i=2}^{k} \left(\left(E_{n-k+2}^{n-k+1}(1) \right)^{u_{n-k+1,n-k+2}} \right)^{\mathrm{T}} \times I \times \prod_{k=2}^{n} \prod_{i=2}^{k} \left(E_k^{k-i+1}(1) \right)^{l_{k,i-1}}$

Illustrative Example

Let

$$T = \begin{bmatrix} 2897 & 3844 & 600 & 48 \\ 3844 & 5441 & 880 & 72 \\ 600 & 880 & 145 & 12 \\ 48 & 72 & 12 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 24 & 48 \\ 0 & 1 & 16 & 72 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 24 & 16 & 1 & 0 \\ 48 & 72 & 12 & 1 \end{bmatrix}$$

The matrix in this example can be factorized as

$$T = \left(\left(E_4^1(1) \right)^{12} \right)^{\mathrm{T}} \times \left(\left(E_4^2(1) \right)^{72} \right)^{\mathrm{T}} \times \left(\left(E_4^3(1) \right)^{48} \right)^{\mathrm{T}} \times \left(\left(E_3^1(1) \right)^{16} \right)^{\mathrm{T}} \times \left(\left(E_3^2(1) \right)^{24} \right)^{\mathrm{T}} \times \left(\left(E_2^1(1) \right)^{4} \right)^{\mathrm{T}} \times I$$

$$\times \left(E_2^1(1) \right)^{4} \times \left(E_3^2(1) \right)^{24} \times \left(E_3^1(1) \right)^{16} \times \left(E_4^3(1) \right)^{48} \times \left(E_4^2(1) \right)^{72} \times \left(E_4^1(1) \right)^{12}$$

Note that the number of the factored matrices equal

$$2 \times (12 + 72 + 48 + 16 + 24 + 4) = 352 \text{ matrix}$$

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