Inverse Nonnegativity of Tridiagonal *M*-Matrices under Diagonal Element-Wise Perturbation

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Abstract

One of the most important properties of *M*-matrices is element-wise non-negative of its inverse. In this paper, we consider element-wise perturbations of tridiagonal *M*-matrices and obtain bounds on the perturbations so that the non-negative inverse persists. The largest interval is given by which the diagonal entries of the inverse of tridiagonal *M*-matrices can be perturbed without losing the property of total nonnegativity. A numerical example is given to illustrate our findings.

Keywords

1. Introduction

In many mathematical problems, Z-matrices and M-matrices play an important role. It is often useful to know the properties of their inverses, especially when the Z-matrices and the M-matrices have a special combinatorial structure, for more details we refer the reader [1]. M-matrices have important applications, for instance, in iterative methods, in numerical analysis, in the analysis of dynamical systems, in economics, and in mathematical programming. One of the most important properties of some kinds of M-matrices is the nonnegativity of their inverses, which plays central role in many of mathematical problems.

An $n \times n$ real matrix $M = \left(m_{ij} \right)$ is called M-matrix if $m_{ii} > 0$, $i = 1, \cdots, n$ and $m_{ij} \leq 0$, $i \neq j$, over the

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years, *M*-matrices have considerable attention, in large part because they arise in many applications [2] [3]. Recently, a noticeable amount of attention has turned to the inverse of tridiagonal *M*-matrices (those matrices which happen to be inverses of special form of *M*-matrices with property $a_{ij} = 0$ whenever $|i - j| > 1$) and M is generalized strictly diagonally dominant. A matrix is said to be generalized (strictly) diagonally dominant if $m_{ii} > \sum\limits_{\substack{i=1 \\ i \neq j}}^{n} m_{ij}$. Of particular importance to us is the fact that since M is an *M*-matrix it is non-singular and

$M^{-1} > 0$ where the inequality is satisfied element-wise. A rich class of *M*-matrices were introduced by Ostrowski in 1937 [4], with reference to the work of Minkowski [5] [6]. A condition which is easy to check is that a matrix M is an *M*-matrix if and only if $m_{ij} \leq 0$, $i \neq j$ and $m_{ii} > 0$, and M is generalized strictly diagonally dominant.

In this paper, we consider the inverse of perturbed *M*-matrix. Specifically we consider the effect of changing single elements inside the diagonal of M^{-1}. We are interested in the large amount by which the single diagonal element of M^{-1} can be varied without losing the property of total nonnegativity.

The reminder of the paper is organized as follows. In section 2, we explain our notations and some needed important definitions are presented. In section 3, some auxiliary results and important prepositions and lemmas are stated. In section 4, we present our results.

2. Notations

In this section we introduce the notation that will be used in developing the paper. For k, n we denote by $Q_{k,n}$ the set of all strictly increasing sequences of k integers chosen from $\{1, 2, \cdots, n\}$. For $\alpha = \{\alpha_1, \cdots, \alpha_k\}$, $\beta = \{\beta_1, \cdots, \beta_k\} \in Q_{k,n}$, we denote by $A[\alpha|\beta]$ the $k \times k$ submatrix of A contained in the rows indexed by $\alpha_1, \cdots, \alpha_k$ and columns indexed by $\beta_1, \cdots, \beta_k$. A matrix A is called totally positive (abbreviated TP henceforth) if $\det A[\alpha|\beta] > 0$ and totally nonnegative (abbreviated TN) if $\det A[\alpha|\beta] \geq 0$ for all $\alpha, \beta \in Q_{k,n}$, $k = 1, 2, \cdots, n$. For a given index $\alpha = \{\alpha_1, \cdots, \alpha_k\}$, with property $\alpha_j < \alpha_{j+1}$, $j = 1, \cdots, k-1$, the dispersion of α , denoted by $d(\alpha)$, is defined to be $d(\alpha) = \sum\limits_{j=1}^{k-1} \alpha_{j+1} - \alpha_j - 1 = \alpha_k - \alpha_1 - k + 1$.

Throughout this paper we use the following notation for general tridiagonal *M*-matrix:

$$
M = \begin{bmatrix}
a_1 & -b_1 & 0 & \cdots & & 0 \\
-c_1 & a_2 & -b_2 & & & \vdots \\
0 & \ddots & \ddots & \ddots & & 0 \\
\vdots & & -c_{n-2} & a_{n-1} & -b_{n-1} \\
0 & \cdots & 0 & -c_{n-1} & a_n
\end{bmatrix}
$$

where a_i , b_i and $c_i > 0$, and each a_i is large enough that M is strictly diagonally dominant.

We let E_{ij} to be the square standard basis matrix whose only nonzero entry is 1 that occurs in the (i, j) position.

Definition 2.1 Compound Matrices ([7], p. 19).

Let $A = (a_{ij})_{i,j=1}^{n}$ be a square matrix of order n . Let $\alpha = \{1, \cdots, n\}$ be the index set of cardinality n , defining $\alpha_i = \alpha \setminus \{n-i+1\}$, $i = \{1, \cdots, n\}$ are the index sets of cardinality $n-1$.

Construct the following table which depends on α_i .

	α_1	α_2	$\cdots$	α_n
α_1	$\det A(\alpha_1, \alpha_1)$	$\det A(\alpha_1, \alpha_2)$	$\cdots$	$\det A(\alpha_1, \alpha_n)$
α_2	$\det A(\alpha_2, \alpha_1)$	$\det A(\alpha_2, \alpha_2)$	$\cdots$	$\det A(\alpha_2, \alpha_n)$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
α_n	$\det A(\alpha_n, \alpha_1)$	$\det A(\alpha_n, \alpha_2)$	$\cdots$	$\det A(\alpha_n, \alpha_n)$

$$C_{n-1}(A) = \begin{bmatrix} \det A(\alpha_1,\alpha_1) & \det A(\alpha_1,\alpha_2) & \cdots & \det A(\alpha_1,\alpha_n) \\ \det A(\alpha_2,\alpha_1) & \det A(\alpha_2,\alpha_2) & \cdots & \det A(\alpha_2,\alpha_n) \\ \vdots & \vdots & \cdots & \vdots \\ \det A(\alpha_n,\alpha_1) & \det A(\alpha_n,\alpha_2) & \cdots & \det A(\alpha_n,\alpha_n) \end{bmatrix}$$

The created matrix is called $(n-1)^{th}$, compound matrix of A.

For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$ with indexed sets $\alpha_1 = \{1,2\}$, $\alpha_2 = \{1,3\}$ and $\alpha_3 = \{2,3\}$.

Then $C_2(A) = \begin{bmatrix} \det A(\alpha_1,\alpha_1) & \det A(\alpha_1,\alpha_2) & \det A(\alpha_1,\alpha_3) \\ \det A(\alpha_2,\alpha_1) & \det A(\alpha_2,\alpha_2) & \det A(\alpha_2,\alpha_3) \\ \det A(\alpha_3,\alpha_1) & \det A(\alpha_3,\alpha_2) & \det A(\alpha_3,\alpha_3) \end{bmatrix} = \begin{bmatrix} -3 & -6 & -3 \\ -6 & -12 & -6 \\ -3 & -6 & -3 \end{bmatrix}$.

3. Auxiliary Results

We start with some basic facts on tridiagonal M-matrices. We can find the determinant of any $n \times n$ tridiagonal M-matrix $M = (m_{ij})$ by using the following recursion equation [8] [9].

$$\det M = a_1 \det M(2,\cdots,n) - b_1 c_1 \det M(3,\cdots,n)$$
$$= a_n \det M(1,\cdots,n-1) - b_{n-1}c_{n-1}\det M(1,\cdots,n-2).$$

And we have the following proposition for finding the determinant of a $n \times n$ tridiagonal M-matrix $M = (m_{ij})$.

Proposition 3.1 ([10], formula 4.1) For any $n \times n$ tridiagonal M-matrix $M = (m_{ij})$ the following relation is true

$$\det M = \det M(1,\cdots,i-1)\det M(i+1,\cdots,n) - b_i c_i \det M(1,\cdots,i-2)\det M(i+1,\cdots,n), \quad i = 2,\cdots,n.$$

We will present now some of propositions of nonsingular totally nonnegative matrices which important for our work.

Proposition 3.2 [10] [11]

For any nonsingular totally nonnegative matrix $A = (a_{ij})_{i,j=1}^n$, all principle minors are positive.

That is $\det A(\alpha) > 0$, for all k and $\alpha \in Q_{n,k}$.

Proposition 3.3 ([7], p. 21)

Let M be a nonsingular tridiagonal M-matrix, and M^{-1} be the inverse of the matrix M then

$$\det M^{-1} = \frac{\det M^{-1}(\alpha)}{\det M(\alpha')}, \text{ when } d(\alpha) = 0.$$

In the sequel we will make use the following lemma, see, e.g. [12].

Lemma 3.4 (Sylvester Identity)

Partition square matrix p of order n, $n > 2$, as:

$$P = \begin{bmatrix} a & P_{12} & b \\ P_{21} & P_{22} & P_{23} \\ c & P_{32} & d \end{bmatrix},$$

where P_{22} square matrix of order $n-2$ and a, b, c and d are scalars.

Define the submatrices

$$A = \begin{bmatrix} a & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, B = \begin{bmatrix} P_{12} & b \\ P_{22} & P_{23} \end{bmatrix}$$

$$C = \begin{bmatrix} P_{21} & P_{22} \\ c & P_{32} \end{bmatrix}, D = \begin{bmatrix} P_{22} & P_{23} \\ P_{32} & d \end{bmatrix}$$

If P_{22} is nonsingular, then

$$\det P = \frac{\det A \det D - \det B \det C}{\det P_{22}}$$

Lemma 3.5 ([11], p.199) Let $A = (a_{ij})$ be a square matrix of order n, with $\det A(2,\cdots,n) \neq 0$. Then $A - xE_{11}$ is totally nonnegative $\forall x \in \left[0, \dfrac{\det A}{\det A(2,\cdots,n)}\right]$.

We now state an important result which links the determinant of M-matrix with the value of the elements of its inverse.

Lemma 3.6 [10] Let $M = \left[m_{ij}\right]_{i,j=1}^{n}$ be a tridiagonal matrix of order n, then we can find the elements of inverse matrix $M^{-1} = \left[m_{ij}^{-1}\right]_{i,j=1}^{n}$ by using the following formula

$$m_{ij}^{-1} = (-1)^{i+j} \frac{\det M\left(1,2,\cdots,\hat{j},\cdots,n/1,2,\cdots,\hat{i},\cdots,n\right)}{\det M}.$$

4. Main Results

In this section, we present our results based on the inverse of tridiagonal M-matrices. Firstly we begin with the following theorem.

Theorem 4.1

Let $M = \begin{bmatrix} a_1 & -b_1 & 0 & \cdots & 0 \\ -c_1 & a_2 & -b_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -c_{n-2} & a_{n-1} & -b_{n-1} \\ 0 & \cdots & 0 & -c_{n-1} & a_n \end{bmatrix}$ be strictly diagonally dominant M-matrix.

If $C_{n-1}(M)$ is the $(n-1)^{th}$ compound matrix of M then the matrix $S\left(C_{n-1}(M)\right)^{\mathrm{T}} S$ is totally nonnegative matrix. Moreover, $\det S\left(C_{n-1}(M)\right)^{\mathrm{T}} S = (\det M)^{n-1}$ where $S = \mathrm{diag}\left(1,-1,\cdots,(-1)^{n+1}\right)$

Proof: Let M be strictly diagonally dominant M-matrix.

Then $M^{-1} = \dfrac{S\left(C_{n-1}(M)\right)^{\mathrm{T}} S}{\det M}$ is totally nonnegative matrix. So is $S\left(C_{n-1}(M)\right)^{\mathrm{T}} S$.

You can find this formula in ([7], p. 21).

There is an explicit formula for the determinant of $\det S\left(C_{n-1}(M)\right)^{\mathrm{T}} S = (\det M)^{n-1}$ given as

$$\det S\left(C_{n-1}(M)\right)^{\mathrm{T}} S = \begin{vmatrix} \det M(\alpha_1,\alpha_1) & -\det M(\alpha_2,\alpha_1) & \cdots & (-1)^{n+1}\det M(\alpha_n,\alpha_1) \\ -\det M(\alpha_1,\alpha_2) & \det M(\alpha_2,\alpha_2) & \cdots & (-1)^{n+2}\det M(\alpha_n,\alpha_2) \\ \vdots & \vdots & \cdots & \vdots \\ (-1)^{n+1}\det M(\alpha_1,\alpha_2) & (-1)^{n+2}\det M(\alpha_2,\alpha_n) & \cdots & (-1)^{2n}\det M(\alpha_n,\alpha_n) \end{vmatrix}$$

$$= \det A(\alpha_1,\alpha_1) \begin{vmatrix} 1 & -\dfrac{\det M(\alpha_2,\alpha_1)}{\det M(\alpha_1,\alpha_1)} & \cdots & (-1)^{n+1}\dfrac{\det M(\alpha_n,\alpha_1)}{\det M(\alpha_1,\alpha_1)} \\ -\det M(\alpha_1,\alpha_2) & \det M(\alpha_2,\alpha_2) & \cdots & (-1)^{n+2}\det M(\alpha_n,\alpha_2) \\ \vdots & \vdots & \cdots & \vdots \\ (-1)^{n+1}\det M(\alpha_1,\alpha_n) & (-1)^{n+2}\det M(\alpha_2,\alpha_n) & \cdots & \det M(\alpha_n,\alpha_n) \end{vmatrix}.$$

Multiply the first row by $(-1)^i \det A(\alpha_1, \alpha_i)$ and add it to the ith row to obtain

$$= \det M(\alpha_1, \alpha_1) \begin{vmatrix} 1 & -\dfrac{\det M(\alpha_2, \alpha_1)}{\det M(\alpha_1, \alpha_1)} & \cdots & (-1)^{n+1}\dfrac{\det M(\alpha_n, \alpha_1)}{\det M(\alpha_1, \alpha_1)} \\ 0 & \dfrac{\det M(\alpha_2, \alpha_2)\det M(\alpha_1, \alpha_1) - \det M(\alpha_2, \alpha_1)\det M(\alpha_1, \alpha_2)}{\det M(\alpha_1, \alpha_1)} & \cdots & \dfrac{(-1)^{n+2}\det M(\alpha_n, \alpha_2)\det M(\alpha_1, \alpha_1) + (-1)^{n+1}\det M(\alpha_n, \alpha_1)\det M(\alpha_1, \alpha_2)}{\det A(\alpha_1, \alpha_1)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \dfrac{(-1)^{n+2}\det M(\alpha_2, \alpha_n)\det M(\alpha_1, \alpha_1) + (-1)^{n+1}\det M(\alpha_1, \alpha_n)\det M(\alpha_2, \alpha_1)}{\det M(\alpha_1, \alpha_1)} & \cdots & \dfrac{\det M(\alpha_n, \alpha_n)\det M(\alpha_1, \alpha_1) + (-1)^{2n+1}\det M(\alpha_1, \alpha_n)\det M(\alpha_n, \alpha_1)}{\det M(\alpha_1, \alpha_1)} \end{vmatrix}$$

$$= \frac{1}{(\det M(\alpha_1, \alpha_1))^{n-1}} \begin{vmatrix} 1 & -\dfrac{\det M(\alpha_2, \alpha_1)}{\det M(\alpha_1, \alpha_1)} & \cdots & (-1)^{n+1}\dfrac{\det M(\alpha_n, \alpha_1)}{\det M(\alpha_1, \alpha_1)} \\ 0 & \det M(\alpha_2, \alpha_2)\det M(\alpha_1, \alpha_1) - \det M(\alpha_2, \alpha_1)\det M(\alpha_1, \alpha_2) & \cdots & (-1)^{n+2}\det M(\alpha_n, \alpha_2)\det M(\alpha_1, \alpha_1) + (-1)^{n+1}\det M(\alpha_n, \alpha_1)\det M(\alpha_1, \alpha_2) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & (-1)^{n+2}\det M(\alpha_2, \alpha_n)\det M(\alpha_1, \alpha_1) + (-1)^{n+1}\det M(\alpha_1, \alpha_n)\det M(\alpha_2, \alpha_1) & \cdots & \det M(\alpha_n, \alpha_n)\det M(\alpha_1, \alpha_1) + (-1)^{2n+1}\det M(\alpha_1, \alpha_n)\det M(\alpha_n, \alpha_1) \end{vmatrix}$$

$$= \frac{1}{(\det M(\alpha_1, \alpha_1))^{n-1}} \begin{vmatrix} 1 & -\dfrac{\det M(\alpha_2, \alpha_1)}{\det M(\alpha_1, \alpha_1)} & \cdots & (-1)^{n+1}\dfrac{\det M(\alpha_n, \alpha_1)}{\det M(\alpha_1, \alpha_1)} \\ 0 & \det M((\alpha_1)_1, (\alpha_1)_1)\det M & \cdots & (-1)^n \det M((\alpha_1)_1, (\alpha_1)_{n-1})\det M \\ \vdots & \vdots & \cdots & \vdots \\ 0 & (-1)^n \det M((\alpha_1)_1, (\alpha_1)_{n-1})\det M & \cdots & \det M((\alpha_1)_{n-1}, (\alpha_1)_{n-1})\det M \end{vmatrix}.$$

where $(\alpha_1)_i = \alpha_1 \setminus \{n-i\} = \dfrac{(\det M)^{n-1}}{(\det A(\alpha_1, \alpha_1))^{n-1}} \begin{vmatrix} 1 & -\dfrac{\det M(\alpha_2, \alpha_1)}{\det M(\alpha_1, \alpha_1)} & \cdots & (-1)^{n+1}\dfrac{\det M(\alpha_n, \alpha_1)}{\det A(\alpha_1, \alpha_1)} \\ 0 & \det M((\alpha_1)_1, (\alpha_1)_1) & \cdots & (-1)^n \det M((\alpha_1)_1, (\alpha_1)_{n-1}) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & (-1)^n \det M((\alpha_1)_1, (\alpha_1)_{n-1}) & \cdots & \det M((\alpha_1)_{n-1}, (\alpha_1)_{n-1}) \end{vmatrix},$

And now apply an induction argument to get the result.

Numerical Example: Let $M = \begin{bmatrix} 3 & -1 & 0 \\ -2 & 5 & -1 \\ 0 & -2 & 3 \end{bmatrix}$ be strictly diagonally dominant M-matrix, then

$C_2(M) = \begin{bmatrix} 13 & -3 & 1 \\ -6 & 9 & -3 \\ 4 & -6 & 13 \end{bmatrix}$ and $S(C_2(M))^T S = \begin{bmatrix} 13 & 6 & 4 \\ 3 & 9 & 6 \\ 1 & 3 & 13 \end{bmatrix}$ is totally nonnegative.

Note that $\det(M) = 33$ and $\det(S(C_2)^T S) = 33^2 = 1089$

Numerically we can conclude the following fact.

Fact: For any $n \times n$ tridiagonal M-matrix $M = (m_{ij})$ the following formula is true.

$$\det M^{-1}(1, \cdots, n-1/2, \cdots, n) = \det M^{-1}(2, \cdots, n/1, \cdots, n-1) = 0 \quad \text{for} \quad n \geq 3$$

Moreover, $\det M^{-1} = \dfrac{1}{\det M}$

To prove this result we use **Theorem 4.1**.

Suppose M is nonsingular then $M^{-1} = \dfrac{S(C_{n-1}(M))^T S}{\det M}$, so

$$\det M^{-1} = \det\left(\frac{S(C_{n-1}(M))^T S}{\det M}\right) = \frac{\det(S(C_{n-1}(M))^T S)}{(\det M)^n} = \frac{(\det M)^{n-1}}{(\det M)^n} = \frac{1}{\det M}$$

For example, when $n = 3$, the M-matrix of our form

$$M = \begin{bmatrix} a_1 & -b_1 & 0 \\ -c_1 & a_2 & -b_2 \\ 0 & -c_2 & a_3 \end{bmatrix} \text{ has an inverse given as}$$

$$M^{-1} = \begin{bmatrix} m_{11}^{-1} & m_{12}^{-1} & m_{12}^{-1} \\ m_{21}^{-1} & m_{22}^{-1} & m_{23}^{-1} \\ m_{31}^{-1} & m_{32}^{-1} & m_{33}^{-1} \end{bmatrix} = \begin{bmatrix} \dfrac{\det M\,(2,3/2,3)}{\det M} & -\dfrac{\det M\,(1,3/2,3)}{\det M} & \dfrac{\det M\,(1,2/2,3)}{\det M} \\ -\dfrac{\det M\,(2,3/1,3)}{\det M} & \dfrac{\det M\,(1,3/1,3)}{\det M} & -\dfrac{\det M\,(1,2/1,3)}{\det M} \\ \dfrac{\det M\,(2,3/1,2)}{\det M} & -\dfrac{\det M\,(1,3/1,2)}{\det M} & \dfrac{\det M\,(1,2/1,2)}{\det M} \end{bmatrix}$$

$$\det M^{-1}(1,2/2,3) = \left(-\frac{\det M\,(1,3/2,3)}{\det M}\right)\left(-\frac{\det M\,(1,2/1,3)}{\det M}\right) - \left(\frac{\det M\,(1,2/2,3)}{\det M}\right)\left(\frac{\det M\,(1,3/1,3)}{\det M}\right)$$

$$= (b_1 a_3)(a_1 b_2) - (b_1 b_2)(a_1 a_3) = 0.$$

Similarly we can find $\det M^{-1}(2,3/1,2) = 0$.

Illustrative Example: Let $M = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 4 & -2 \\ 0 & -3 & 5 \end{bmatrix}$ be a tridiagonal M-matrix and

$$M^{-1} = \begin{bmatrix} 0.3784 & 0.1351 & 0.0541 \\ 0.1351 & 0.4054 & 0.1622 \\ 0.0811 & 0.2432 & 0.2973 \end{bmatrix}$$

Note that $\det M^{-1}(1,2/2,3) = \det M^{-1}(2,3/1,2) \cong 0$

Observe that the error came from the rounded to the nearest part of 10,000.

Theorem 4.2 Let M be a strictly diagonally dominant M-matrix, if $\alpha = \{1,\cdots,n-1\}$, $\beta = \{2,\cdots,n\}$, $\gamma = \alpha \cap \beta$ then

$$\det M^{-1} = \frac{\det M^{-1}[\alpha|\alpha] \times \det M^{-1}[\beta|\beta]}{\det M^{-1}(\gamma)}$$

Proof:

$$\text{Assume } M^{-1} = \begin{bmatrix} a & m_{12}^{-1} & b \\ m_{21}^{-1} & M_{22}^{-1} & m_{23}^{-1} \\ c & m_{32}^{-1} & d \end{bmatrix} \text{ and}$$

$$M^{-1}[\alpha|\alpha] = \begin{bmatrix} a & m_{12}^{-1} \\ m_{21}^{-1} & M_{22}^{-1} \end{bmatrix}, \quad M^{-1}[\alpha|\beta] = \begin{bmatrix} m_{12}^{-1} & b \\ M_{22}^{-1} & m_{23}^{-1} \end{bmatrix}, \quad M^{-1}[\beta|\alpha] = \begin{bmatrix} m_{21}^{-1} & M_{22}^{-1} \\ c & m_{32}^{-1} \end{bmatrix}, \quad M^{-1}[\beta|\beta] = \begin{bmatrix} M_{22}^{-1} & m_{23}^{-1} \\ m_{32}^{-1} & d \end{bmatrix}.$$

Note that $M^{-1}[\alpha|\beta] = M^{-1}[\beta|\alpha] = 0$ by previous fact, and by using Sylvester's identity, we have

$$\det M^{-1} = \frac{\det M^{-1}[\alpha|\alpha] \times \det M^{-1}[\beta|\beta] - \det M^{-1}[\alpha|\beta] \times \det M^{-1}[\beta|\alpha]}{\det M_{22}^{-1}}$$

$$= \frac{\det M^{-1}[\alpha|\alpha] \times \det M^{-1}[\beta|\beta]}{\det M_{22}^{-1}}.$$

Moreover we conclude the following theorem.

Theorem 4.3 Let M be the M-matrix defined above then

$$\det\left(M^{-1}-xE_{ii}\right)=\det M^{-1}-\frac{x\times a_i}{\det M}$$

For example

$$M=\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -3 & 5 \end{bmatrix},\quad M^{-1}=\begin{bmatrix} 0.6308 & 0.2615 & 0.0769 & 0.0154 \\ 0.2615 & 0.5231 & 0.1538 & 0.0308 \\ 0.1538 & 0.3077 & 0.3846 & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & 0.2462 \end{bmatrix}$$

Let $x=0.2$ then $M^{-1}+0.2E_{11}=\begin{bmatrix} 0.4308 & 0.2615 & 0.0769 & 0.0154 \\ 0.2615 & 0.5231 & 0.1538 & 0.0308 \\ 0.1538 & 0.3077 & 0.3846 & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & 0.2462 \end{bmatrix}$

$$\det\left(M^{-1}+0.2E_{11}\right)=0.0092 \quad\text{and}\quad \det M^{-1}-\frac{x\times a_1}{\det M}=0.0154-\frac{0.2\times 2}{65}=0.0092$$

Now, we will perturb elements inside the diagonal band of the inverse of M-matrix without losing the nonnegativity property. We begin with the $(1,1)$ element then generalize to other elements.

Theorem 4.4 Let M be a strictly diagonally dominant tridiagonal M-matrix. Then the matrix $M^{-1}-xE_{11}$ is totally nonnegative for all $x\in\left[0,\dfrac{1}{a_1}\right]$.

Proof:

$$\text{Let}\quad M=\begin{bmatrix} a_1 & -b_1 & 0 & \cdots & 0 \\ -c_1 & a_2 & -b_2 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & -c_{n-2} & a_{n-1} & -b_{n-1} \\ 0 & \cdots & 0 & -c_{n-1} & a_n \end{bmatrix}$$

Be a nonsingular strictly diagonally dominant tridiagonal M-matrix then M^{-1} is totally nonnegative.

By **Lemma 3.5** and **Proposition 3.2**, we have

$M^{-1}-xE_{11}$ is totally nonnegative $\forall x\in\left[0,\dfrac{\det M^{-1}}{\det M^{-1}\{2,\cdots,n\}}\right]$.

By using the formula in **Proposition 3.3**

$$\frac{\det M^{-1}}{\det M^{-1}\{2,\cdots,n\}}=\frac{(-1)^{1+1}\dfrac{\det M^{-1}\{2,\cdots,n\}}{a_1}}{\det M^{-1}\{2,\cdots,n\}}=\frac{1}{a_1}.$$

Note that a similar result holds for decreasing the element m_{nn}^{-1} by considering the matrix $\tilde{S}M^{-1}\tilde{S}$, which reverses the matrix M^{-1} as the relation $m_{ij}^{-1}=m_{n-i+1,n-j+1}^{-1}$.

We can generalize this result for the other elements of diagonal.

Theorem 4.5 Assume M is a strictly diagonally dominant tridiagonal M-matrix. Then the matrix $M^{-1}-xE_{ii}$ is totally nonnegative for all $x\in\left[0,\dfrac{1}{a_i}\right]$.

Proof: Suppose that $X=M^{-1}-xE_{ii}$ is not totally nonnegative for all $x\in\left[0,\dfrac{1}{a_i}\right]$, then there exist $\alpha,\beta\in Q_{n,k}$ both contain i such that $\det X=\det\left(\left(M^{-1}-xE_{ii}\right)(\alpha,\beta)\right)<0$.

To compute $\det X$ expand the determinant along the ith row of X then

$\det X = (-1)^{\alpha_i + \beta_i} (-x) \det X (\{\alpha_i\}, \{\beta_i\}) + \det M^{-1} (\alpha', \beta')$ where $\alpha' \subset \alpha$, $\beta' \subset \beta$, and $\det M^{-1} (\alpha', \beta')$ is some minor of M^{-1}.

Take the case when $\alpha_i + \beta_i$ odd. Thus, $\det X$ is a positive linear compination of minors of M^{-1} and hence is positive, which contradicts the assumption.

Now suppose $\exists x \notin \left[0, \dfrac{1}{a_i}\right]$ such that $M^{-1} - xE_{ii}$ is totally nonnegative matrix.

Suppose that $x > \dfrac{1}{a_i}$, then by **Theorem 4.3**.

$$\det\left(M^{-1} + xE_{ii}\right) = \det M^{-1} - \frac{x \times a_i}{\det M} < \det M^{-1} - \frac{1}{\det M} = 0 \text{, since } \det M^{-1} = \frac{1}{\det M}$$

which contradicts the nonnegativity of $M^{-1} - xE_{ii}$.

Numerical Example: Let $M = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -3 & 5 \end{bmatrix}$ is strictly diagonally dominant tridiagonal M-matrix

$$M^{-1} = \begin{bmatrix} 0.6308 & 0.2615 & 0.0769 & 0.0154 \\ 0.2615 & 0.5231 & 0.1538 & 0.0308 \\ 0.1538 & 0.3077 & 0.3846 & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & 0.2462 \end{bmatrix} \text{ so}$$

The matrices

$$M^{-1} + xE_{11} = \begin{bmatrix} [0.1308, 0.6308] & 0.2615 & 0.0769 & 0.0154 \\ 0.2615 & 0.5231 & 0.1538 & 0.0308 \\ 0.1538 & 0.3077 & 0.3846 & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & 0.2462 \end{bmatrix}, \ x \in [0, 0.5],$$

$$M^{-1} + xE_{22} = \begin{bmatrix} 0.6308 & 0.2615 & 0.0769 & 0.0154 \\ 0.2615 & [0.1898, 0.5231] & 0.1538 & 0.0308 \\ 0.1538 & 0.3077 & 0.3846 & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & 0.2462 \end{bmatrix}, \ x \in [0, 0.3333],$$

$$M^{-1} + xE_{33} = \begin{bmatrix} 0.6308 & 0.2615 & 0.0769 & 0.0154 \\ 0.2615 & 0.5231 & 0.1538 & 0.0308 \\ 0.1538 & 0.3077 & [0.1346, 0.3846] & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & 0.2462 \end{bmatrix}, \ x \in [0, 0.25], \text{ and}$$

$$M^{-1} = \begin{bmatrix} 0.6308 & 0.2615 & 0.0769 & 0.0154 \\ 0.2615 & 0.5231 & 0.1538 & 0.0308 \\ 0.1538 & 0.3077 & 0.3846 & 0.0769 \\ 0.0923 & 0.1846 & 0.2308 & [0.0462, 0.2462] \end{bmatrix}, \ x \in [0, 0.2].$$

are TNN matrices.

Note that $\det M^{-1} (\{1, 2, 3\} \setminus \{2, 3, 4\}) = \det M^{-1} (\{2, 3, 4\} \setminus \{1, 2, 3\}) = 0$.

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